



# Sophomore Physics Laboratory (PH005/105)

## Analog Electronics Resonant Circuits

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# Chapter 2

## Resonant Circuits

### 2.1 Introduction

Resonators, one of the most useful and used device, are essentially physical systems that present a more or less pronounced peak in their transfer function.

In general, their performance is measured by a dimensionless parameter named quality factor  $Q$ , which characterizes the sharpness of the resonant peak. The higher the quality factor the sharper is the peak and the better is the resonator.

Quite often, the major issues of building a resonator are to obtain very high quality factors and good stability. For example, mechanical oscillators made of fused silica fibers under load, can achieve quality factors above  $10^8$  in the acoustic band[?]. Very high quality factors in electronics can be achieved using the mechanical resonances of piezoelectric materials such as quartz. Lasers and resonant cavities made of mirrors can be used to build resonators in the optical frequency range. The same principle can be applied in the microwave range. Thermal stabilization is always a key ingredient to obtain high stability.

Resonators made with electronic passive components, reaching quality factors values up to 10-100 or more, are quite easy to realize. In the next sections we will study two typical resonant circuits, the LCR series and LCR parallel circuits.

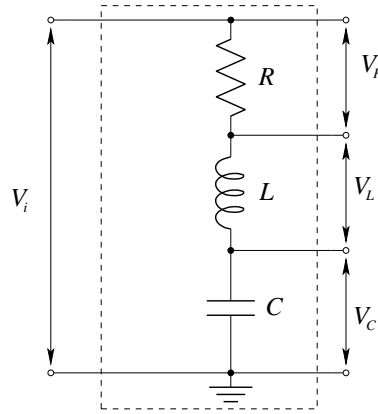


Figure 2.1: LCR series circuit.

## 2.2 The LCR Series Resonant Circuit

Figure 2.1 shows the so called *LCR series resonant circuit*. Depending on voltage difference, we are considering as the circuit output ( the capacitor, the resistor, or the inductor), this circuit shows a different behavior. Let's study indeed in the frequency and in the domain, the response of this passive circuit for each one of the possible non trivial outputs.

### 2.2.1 Frequency Response with Capacitor Voltage Difference as Circuit Output

Considering the voltage difference  $V_C$  across the capacitor the circuit output, we will have

$$V_{in} = \left( R + j\omega L + \frac{1}{j\omega C} \right) I,$$

$$V_C = \frac{1}{j\omega C} I,$$

and the transfer function will be

$$H_C(\omega) = \frac{1}{j\omega RC - \omega^2 LC + 1}.$$

For sake of simplicity it is convenient to define the two following quantities

$$\omega_0^2 = \frac{1}{LC}, \quad Q = \frac{1}{R} \sqrt{\frac{L}{C}} = \omega_0 \frac{L}{R}$$

The parameter  $Q$  is the quality factor of the circuit, and the angular frequency  $\omega_0$  is the resonant frequency of the circuit if  $R = 0$ .

Considering the previous definitions, and after some algebra,  $H_C(\omega)$  becomes

$$H_C(\omega) = \frac{\omega_0^2}{\omega_0^2 - \omega^2 + j\omega \frac{\omega_0}{Q}}. \quad (2.1)$$

Computing the magnitude and phase of  $H_C(\omega)$ , we obtain

$$|H_C(\omega)| = \frac{\omega_0^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + \left(\omega \frac{\omega_0}{Q}\right)^2}},$$

$$\arg[H_C(\omega)] = -\arctan\left(\frac{1}{Q} \frac{\omega_0 \omega}{\omega_0^2 - \omega^2}\right).$$

The magnitude has maximum for

$$\omega_C^2 = \omega_0^2 \left(1 - \frac{1}{2Q^2}\right),$$

and the maximum is

$$|H_C(\omega_C)| = \frac{Q}{\sqrt{1 - \frac{1}{4Q^2}}}.$$

If  $Q \gg 1$  then  $\omega_C \simeq \omega_0$ , and  $|H_C(\omega_C)| \simeq Q$ .

Far from resonance  $\omega_C$ , the approximate behavior of  $|H_C(\omega)|$  is

$$\begin{aligned} \omega \ll \omega_C &\Rightarrow |H_C(\omega)| \simeq 1, \\ \omega \gg \omega_C &\Rightarrow |H_C(\omega)| \simeq \frac{\omega_0^2}{\omega^2}. \end{aligned}$$

Figure 2.2 shows the magnitude and phase of  $H_C(\omega)$ . In this case the circuit is a low pass filter of the second order because of the asymptotic slope  $1/\omega^2$ .

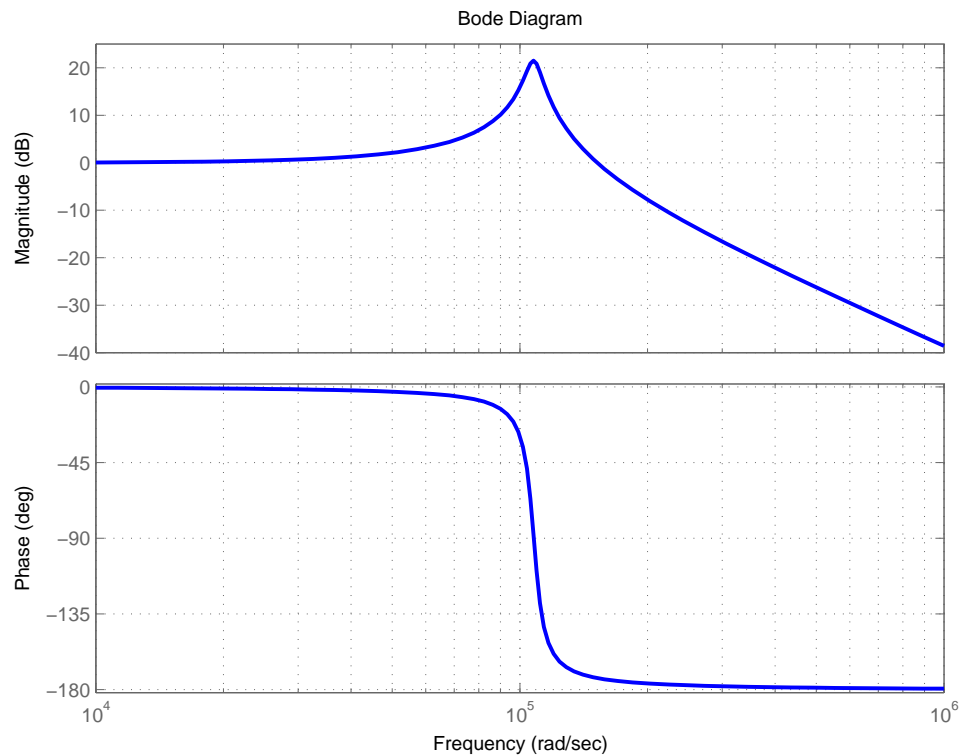


Figure 2.2: Transfer function  $H_C(\omega)$  of the LCR series resonant circuit with a resonant angular frequency  $\omega_C \simeq 10.7\text{krad/s}$ .

### 2.2.2 Frequency Response with Inductor Voltage Difference as Circuit Output

Considering the voltage difference  $V_L$  across the inductor as the circuit output, we will have instead

$$H_L(\omega) = -\frac{\omega^2 LC}{j\omega RC - \omega^2 LC + 1}$$

Using the definition of  $Q$ , and  $\omega_0$  and after some algebra,  $H_L(\omega)$  becomes

$$H_L(\omega) = \frac{-\omega^2}{\omega_0^2 - \omega^2 + j\omega\frac{\omega_0}{Q}} \quad (2.2)$$

Computing the magnitude and phase of  $H_L(\omega)$ , we obtain

$$|H_L(\omega)| = \frac{\omega^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + \left(\omega\frac{\omega_0}{Q}\right)^2}}$$

$$\arg [H_L(\omega)] = \arctan \left( \frac{1}{Q} \frac{\omega\omega_0}{\omega_0^2 - \omega^2} \right)$$

The magnitude has a maximum for

$$\omega_L^2 = \omega_0^2 \frac{1}{1 - \frac{1}{2Q^2}},$$

and the maximum is

$$|H_L(\omega_L)| = \frac{Q}{\sqrt{1 - \frac{1}{4Q^2}}}.$$

If  $Q \gg 1$  then  $\omega_L \simeq \omega_0$ , and  $|H_L(\omega_L)| \simeq Q$ .

Far from resonance  $\omega_L$ , the approximate behavior of  $|H_L(\omega)|$  is

$$\begin{aligned} \omega \ll \omega_L &\Rightarrow |H_L(\omega)| \simeq \frac{\omega^2}{\omega_0^2} \\ \omega \gg \omega_L &\Rightarrow |H_L(\omega)| \simeq 1 \end{aligned}$$

Figure 2.3 shows the magnitude and phase of  $H_L(\omega)$ . In this case the circuit is a second order high pass filter.

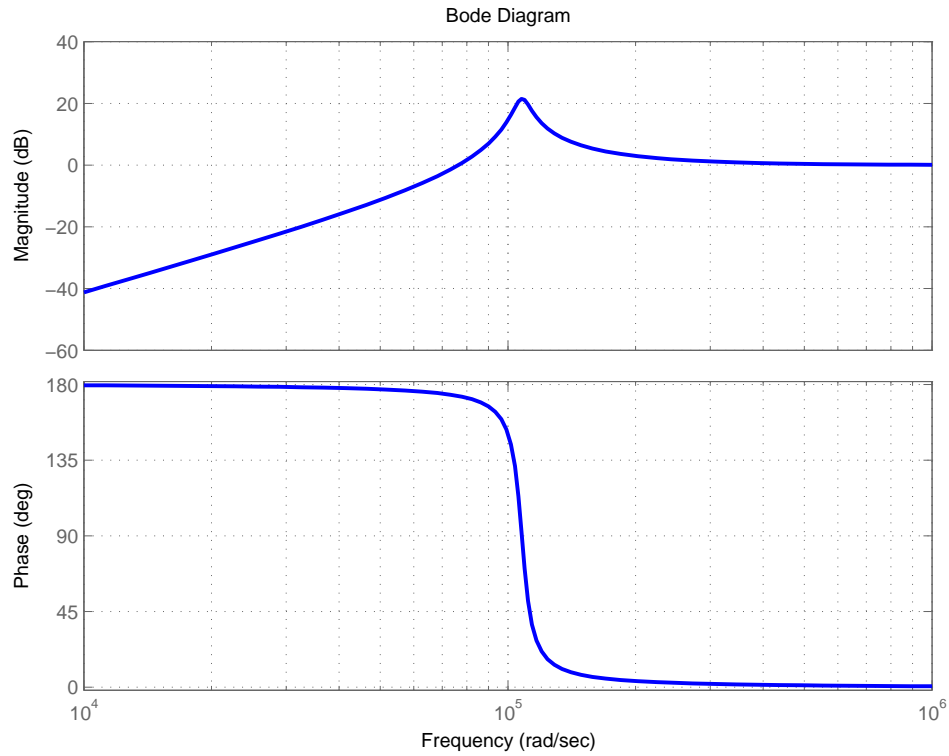


Figure 2.3: Transfer function  $H_L(\omega)$  of the LCR series resonant circuit with a resonant angular frequency  $\omega_L \simeq 10.7\text{krad/s}$ .

### 2.2.3 Frequency Response with the Resistor Voltage Difference as Circuit Output

Considering the voltage difference across the resistor as the circuit output, we will have instead

$$H_R(\omega) = \frac{j\omega RC}{1 - \omega^2 LC + j\omega RC}.$$

Using the definition of  $Q$  and  $\omega_0$ , and after some algebra,  $H_R(\omega)$  becomes

$$H_R(\omega) = \frac{j\omega \frac{\omega_0}{Q}}{\omega_0^2 - \omega^2 + j\omega \frac{\omega_0}{Q}}. \quad (2.3)$$

Computing the magnitude and phase of  $H_R(\omega)$ , we obtain

$$|H_R(\omega)| = \frac{\frac{\omega_0}{Q}\omega}{\sqrt{(\omega_0^2 - \omega^2)^2 + \left(\omega\frac{\omega_0}{Q}\right)^2}}$$

$$\arg [H_R(\omega)] = \arctan \left( Q \frac{\omega_0^2 - \omega^2}{\omega\omega_0} \right)$$

The magnitude has maximum for

$$\omega_R^2 = \omega_0^2,$$

and the maximum is

$$|H_R(\omega_R)| = 1.$$

Far from the resonance  $\omega_R$ , the approximate behavior of  $|H_R(\omega)|$  is

$$\omega \ll \omega_R \quad \Rightarrow \quad |H_R(\omega)| \simeq \frac{1}{Q} \frac{\omega}{\omega_0}$$

$$\omega \gg \omega_R \quad \Rightarrow \quad |H_R(\omega)| \simeq \frac{\omega_0}{\omega}$$

Figure 2.4 shows the magnitude and phase of  $H_R(\omega)$ . In this case the circuit is a first order band pass filter.

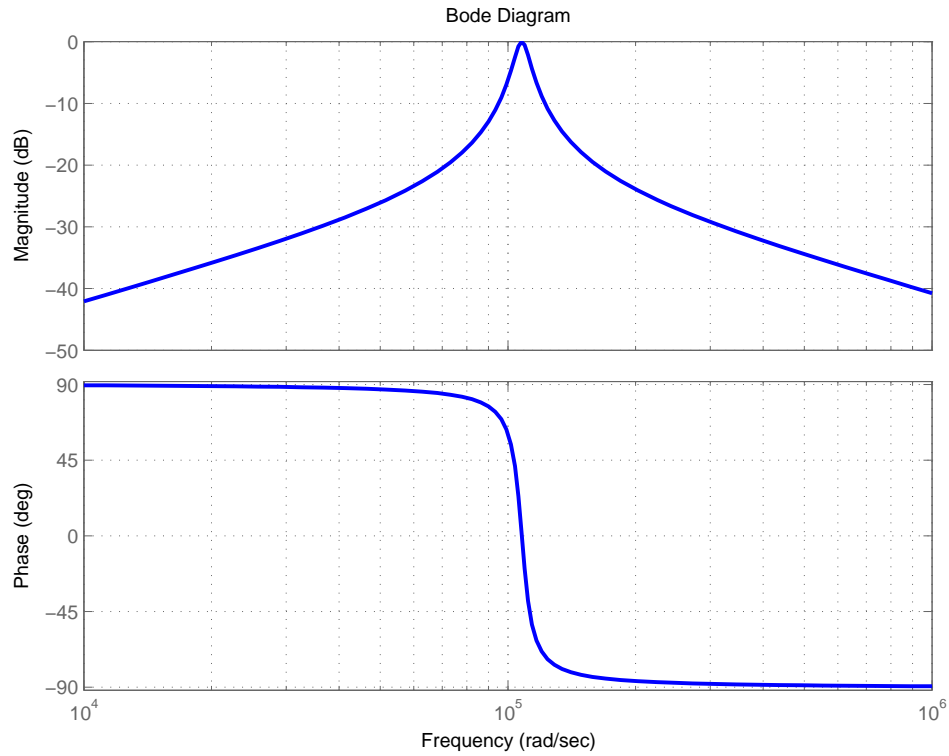


Figure 2.4: Transfer function  $H_R(\omega)$  of the LCR series resonant circuit with resonant angular frequency  $\omega_R \simeq 10.7\text{krad/s}$ .

### 2.2.4 Transient Response

The equation that describes the LCR series circuit response in the time domain is

$$v_i = Ri + L\frac{di}{dt} + \frac{1}{C} \int_0^t i(t')dt', \quad (2.4)$$

where  $i(t)$  is the current flowing through the circuit and  $v_i(t)$  is the input voltage.

Supposing that

$$v_i(t) = \begin{cases} v_0, & t > 0 \\ 0, & t \leq 0 \end{cases},$$

and differentiating both side of eq. 2.4, we obtain the linear differential equation

$$R \frac{di}{dt} + L \frac{d^2i}{dt^2} + \frac{1}{C} i = 0, \quad t > 0$$

or, considering the definition of  $\omega_0$ , and  $Q$ ,

$$\frac{d^2i}{dt^2} + \frac{\omega_0}{Q} \frac{di}{dt} + \omega_0^2 i = 0.$$

The solutions of the characteristic polynomial equation associated with the differential equation are

$$\lambda_{1,2} = -\frac{1}{2} \frac{\omega_0}{Q} \pm \omega_0 \sqrt{\frac{1}{2Q^2} - 1}.$$

As usual, we will have three different solutions depending on the discriminant value

$$\Delta = \frac{1}{2Q^2} - 1.$$

**Under-damped Case: discriminant less than zero** ( $Q > 1/\sqrt{2}$ )

In this case we have two complex conjugate roots and the differential equation solution is the typical exponential ring down

$$i(t) = i_0 e^{-\frac{\omega_0}{2Q}t} \sin(\omega_C t + \varphi_0), \quad \omega_C^2 = \omega_0^2 \left(1 - \frac{1}{2Q^2}\right).$$

**Critically Damped Case: Discriminant equal to zero** ( $Q = 1/\sqrt{2}$ )

In this case we have a critically damped current and no oscillation

$$i(t) = i_0 e^{-\frac{\omega_0}{2Q}t}$$

**Over-damped Case: Discriminant greater than zero** ( $Q < 1/\sqrt{2}$ )

This is the case of two coincident solutions . We will have indeed, an exponential decay (no oscillations)

$$i(t) = i_0 e^{-\frac{\omega_0}{2Q}t} (Ae^{-\omega_c t} + Be^{+\omega_c t}), \quad \omega_c^2 = \omega_0^2 \left(1 - \frac{1}{2Q^2}\right),$$

Voltages across each single element can be easily computed considering the relation between  $v(t)$  and  $i(t)$ .

Let's just write the voltage across the capacitor for the under-damped case. Considering that the integration operation in this case changes just the phase and creates an offset, the voltage across the capacitor, neglecting this offset, will be

$$v_C(t) = v_0 e^{-\frac{\omega_0}{2Q}t} \sin(\omega_c t + \psi).$$

## 2.3 The Tank Circuit or LCR Parallel Circuit.

Figure 2.5 shows the so called *LCR parallel resonant circuit* or *tank circuit*, where the source depicted with an arrow inside a circle is an ideal current source. The resistor of resistance  $r$  accounts for inductor resistance. Let's study the frequency and the transient response using the Thévenin representation shown in figure 2.6.

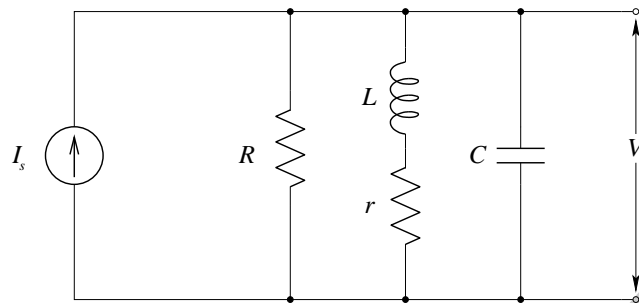


Figure 2.5: The tank circuit.

### 2.3.1 LCR Circuit Frequency Response

Using Thévenin theorem for the current source and  $R$ , the LCR parallel circuit considering the equivalent circuit as shown in figure 2.6 where the

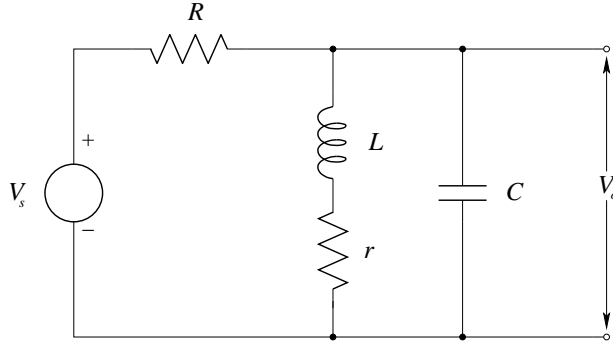


Figure 2.6: The tank circuit with the current source and the resistance  $R$  replaced with the Thévenin equivalent circuit.

current source and the resistor  $R$  have been replaced with the Thévenin circuit.

Considering that the current  $I$  of the current source can be written as

$$I = \frac{V_i}{R'}$$

$$I = Y V_o = \left( \frac{1}{R} + \frac{1}{r + j\omega L} + j\omega C \right) V_o,$$

we have

$$\frac{V_i}{R} = \left( \frac{1}{R} + \frac{1}{r + j\omega L} + j\omega C \right) V_o \quad (2.5)$$

Defining the following complex quantity as

$$\frac{1}{r^*(\omega)} + \frac{1}{j\omega L^*(\omega)} = \frac{1}{r + j\omega L}, \quad (2.6)$$

and

$$R^* = R \parallel r^*,$$

eq. 2.5 becomes

$$\frac{V_i}{R} = \left( \frac{1}{R^*} + \frac{1}{j\omega L^*} + j\omega C \right) V_o$$

After some algebra, we will have

$$\frac{V_o}{V_i} = \frac{j\omega L^*}{R^* - \omega^2 C L^* R^* + j\omega L^* R}. \quad (2.7)$$

Generalizing the definition of  $\omega_0$ , and  $Q$

$$\omega_0^* = \frac{1}{\sqrt{L^*(\omega)C}}, \quad Q^* = R^*(\omega) \sqrt{\frac{C}{L^*(\omega)}},$$

and substituting in eq. 2.7 we finally obtain

$$H(\omega) = \frac{j\omega\omega_0^*/Q^*}{(\omega_0^*)^2 - \omega^2 + j\omega\omega_0^*/Q^*} \frac{R^*}{R}$$

Let's find the implicitly defined functions  $r^*, L^*$ . Using the term containing the inductance  $L$  in eq. 2.6, we obtain

$$\frac{1}{r + j\omega L} = \frac{1}{r \left[1 + \left(\frac{\omega L}{r}\right)^2\right]} + \frac{1}{j\omega L \left[1 + \left(\frac{r}{\omega L}\right)^2\right]}.$$

and finally

$$r^*(\omega) = r \left[1 + \left(\frac{\omega L}{r}\right)^2\right], \quad L^*(\omega) = L \left[1 + \left(\frac{r}{\omega L}\right)^2\right]$$

### 2.3.2 Transfer Function

From the solution of the LCR parallel circuit we have

$$|H(\omega)| = \frac{\frac{\omega\omega_0^*}{Q^*}}{\sqrt{\left[(\omega_0^*)^2 - \omega^2\right]^2 + \left(\frac{\omega\omega_0^*}{Q^*}\right)^2}} \frac{|R^*|}{R}$$

$$\arg(H(\omega)) = \arctan \left\{ Q^* \frac{\omega_0^2 - \omega^2}{\omega\omega_0} \right\},$$

whose bode plots are shown in figure 2.7.

### 2.3.3 Simplest Case

It is worthwhile to notice that if  $r = 0$  we will have much simpler expressions, i.e.

$$\omega_0 = \frac{1}{\sqrt{LC}}, \quad Q = R \sqrt{\frac{C}{L}}.$$

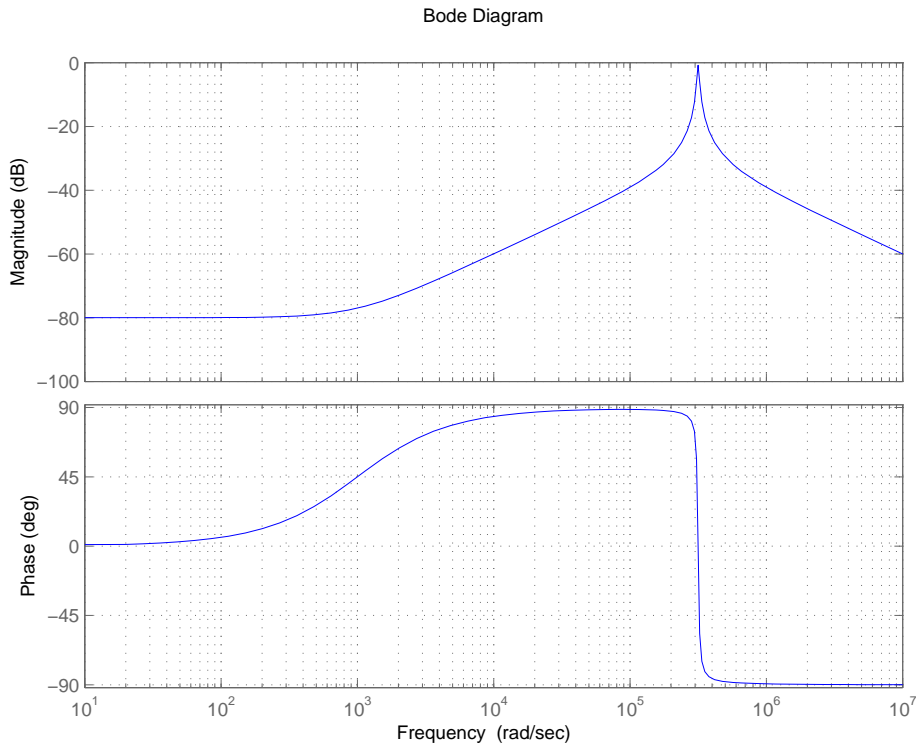


Figure 2.7: Typical bode plot of a LCR parallel circuit with resonant angular frequency near the acoustic band. As expected, if  $r$  is not zero, then the magnitude doesn't go to zero for  $\omega = 0$ .

and

$$H(\omega) = \frac{j\omega\omega_0/Q}{\omega_0^2 - \omega^2 + j\omega\omega_0/Q}$$

The magnitude and the phase will be

$$|H(\omega)| = \frac{\frac{\omega\omega_0}{Q}}{\sqrt{(\omega_0^2 - \omega^2)^2 + \left(\frac{\omega\omega_0}{Q}\right)^2}}$$

$$\arg(H(\omega)) = \arctan \left\{ Q \frac{\omega_0^2 - \omega^2}{\omega\omega_0} \right\},$$

### 2.3.4 High Frequency Approximation

For high frequency  $\omega \gg r/L$ , we have

$$r^*(\omega) \simeq r \left( \omega \frac{L}{r} \right)^2, \quad \Rightarrow \quad L^* \simeq L$$

and  $\omega_0$  becomes

$$\omega_0 \simeq \frac{1}{\sqrt{LC}}.$$

Evaluating the several defined quantities at  $\omega_0$ , we will have

$$\begin{aligned} r^*(\omega_0) &\simeq \frac{L}{rC}, \\ R^*(\omega_0) &\simeq \frac{LR}{RCr + L}, \\ Q^*(\omega_0) &\simeq \frac{LR}{RCr + L} \sqrt{\frac{C}{L}} \end{aligned}$$

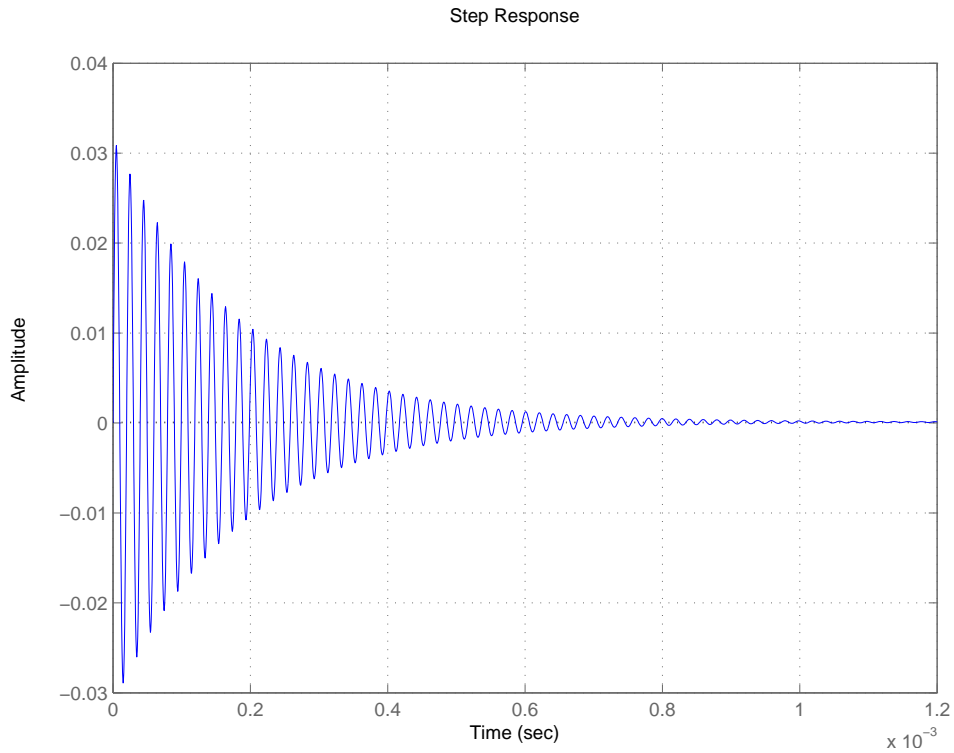


Figure 2.8: Typical step response of a LCR parallel circuit near the acoustic band.

### 2.3.5 LCR Parallel Circuit Transient Response

Let's briefly analyze the response to a step of the LCR parallel circuit for the under-damped case.

If we define the following quantity

$$\gamma = \frac{1}{2Q},$$

called damping coefficient, and if  $0 < \gamma < 1$ , then we will have at the circuit output

$$v(t) = v_0 \frac{2\gamma}{\gamma - 1} e^{-\omega_0 \gamma t} \cos \left( \sqrt{1 - \gamma^2} \omega_0 t + \varphi_0 \right) + v_1.$$

The voltage output  $v(t)$  is a damped sinusoid with angular frequency  $\sqrt{1 - \gamma^2} \omega_0$  and time constant  $\tau = 1/\omega_0 \gamma$ . The DC offset  $v_1$  depends on the inductor resistance  $r$  and the initial step.

Figure 2.8, a typical step response of the LCR circuit shows a ring-down with a DC offset.

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## 2.4 Laboratory Experiment

Real inductors have not negligible resistance. To build a LCR series circuit with a highest quality factor it is indeed necessary to minimize the resistance of the circuit by mounting in series the inductor and the capacitor only. Typical effective resistance of the inductors used in the laboratory is about  $10\Omega$  to  $80\Omega$  at resonance .

Because of the internal resistance of the function generator (the best scenario gives  $\sim 50\Omega$ ) is then comparable at some frequencies to LCR load, we will expect that the approximation of ideal generator will be no longer valid.

Moreover, harmonic distortion of the function generator will be quite evident in the LCR series circuit because of the dependence of the load on the frequency.

An estimation of a ring-down time constant  $\tau$  can be obtained as follows. From the ring-down equation we have that after a time  $t = \tau$  the envelope maximum amplitude is reduced by a factor  $1/3$  ( $e \simeq 1/2.718$ ). This means that we can easily estimate  $\tau$  by just measuring the time needed to reduce the amplitude down to about  $1/3$  of its initial value. Another but quite coarse way is to count how many periods  $n^*$  the amplitude takes to decrease to  $1/3$  of its initial value. Then the estimation will be

$$\tau \simeq Tn^* = \frac{n^*}{\nu_{res}},$$

where  $T$ , and  $\nu_{res}$  are respectively the period and the frequency of the oscillation. Considering that  $Q = \pi\nu_{res}\tau$  then

$$Q \simeq \pi n^* .$$

The quality factor can also be estimated from the frequency response considering that

$$Q = \frac{\nu_{res}}{\Delta\nu},$$

where  $\Delta\nu$  is the Full Width at Half Maximum (FWHM) of the peak resonance.

### 2.4.1 Pre-laboratory Exercises

It is suggested to read the appendix about the electromagnetic noise to complete the pre-lab problems and the laboratory procedure.

1. Determine the capacitance  $C$  of a LCR series circuit necessary to have a resonant frequency  $\nu_C = 20\text{kHz}$  if  $L = 10\text{mH}$ , and  $R = 10\Omega$ . Then, calculate  $Q$ ,  $\tau$ ,  $\nu_0$ , ( $\omega = 2\pi\nu$ ) of the circuit.
2. Find the LCR series input impedance  $Z_i$  and plot its magnitude in a logarithmic scale. Determine at what frequency is the minimum of  $|Z_i|$ .
3. Supposing that the internal resistance of the function generator is  $R_s = 50\Omega$ , and using the previous values for  $L$ ,  $C$ , and  $R$ , calculate the circuit input voltage attenuation at the frequency of  $|Z_i|$  minimum and at twice that frequency.
4. Determine the capacitance  $C$  of a tank circuit necessary to have a resonant frequency  $\nu_C = 20\text{kHz}$  if  $L = 10\text{mH}$ ,  $R = 10\text{k}\Omega$ , and  $r = 10\Omega$ . Use the high frequency approximation. Then, calculate  $Q$ ,  $\tau$ ,  $\nu_0$ , of the circuit.
5. Estimate the time constant  $\tau$  of the ring-down in figure 2.8. Supposing that  $R = 10\text{k}\Omega$ , estimate  $r$  from figure 2.7.
6. Calculate the maximum frequency of the EM field isolated by a Faraday cage with a dimension  $d = 10\text{mm}$  (hint: consult the proper appendix).

### 2.4.2 Procedure

1. Build a LCR series circuit with a resonant frequency of around  $20\text{kHz}$ , using inductance, capacitance, and resistance values calculated in the pre-lab problems. Then, do the following steps:
  - (a) Using the oscilloscope and knowing the expected magnitude and phase values at the resonant frequency  $\nu_C$ , find  $\nu_C$  and compare it with the theoretical value computed using the components measured values.
  - (b) Verify the circuit transfer function  $H_C(\nu)$  using the data acquisition system and the proper software.
  - (c) Estimate the quality factor  $Q$  of the circuit from the transfer function measurement and compare it with the theoretical value.

- (d) Explain why the input voltage  $V_i$  changes in amplitude if we change frequency.
  - (e) Considering the harmonic distortion of the function generator, explain why the frequency spectrum of the input signal changes quite drastically when we approach the resonance  $\nu_C$ .
  - (f) Download the simulation file from the ph5/105 website for the LCR series circuit, input the proper components values, run the AC response simulation, and find the discrepancies between your measurement and the simulation.
  - (g) Modify the simulated circuit to qualitatively account for the eventual notch measured between 100 kHz and 1 MHz (hint: use the inductor model specified in the appendix considering the extra capacitor only).
2. Build a LCR parallel circuit with a resonant frequency around 20kHz, using inductance, capacitance, and resistance values calculated in the pre-lab problems. Then, do the following steps:
- (a) Using the oscilloscope and knowing the expected magnitude and phase values at the resonant frequency  $\nu_C$ , find  $\nu_C$  and compare it with the theoretical value computed using the components measured values.
  - (b) Verify the circuit transfer function  $H_C(\nu)$  using the data acquisition system and the proper software.
  - (c) Estimate the quality factor  $Q$  of the circuit from the step response.
  - (d) Download the simulation file from the ph5/105 website for the LCR parallel circuit, input the proper components values, run the AC response simulation, and find the discrepancies between your measurement and the simulation.
  - (e) Modify the simulated circuit to qualitatively account for the eventual notch measured between 100 kHz and 1MHz (hint: use the capacitor model specified in the appendix considering the extra inductors only).

3. Check the effect of the Faraday cage ( a metallic coffee can) using 10x probe connected to the oscilloscope. Add a 1m long wire to increase the antenna effect.

Note the differences when the antenna is approached to the fluorescent lights, and when you touch the antenna.

Keeping the cage in the same position and without touching the cage, explain what you observe and coarsely estimate the amplitude and frequency content of the picked-up signal in the following conditions:

- (a) Antenna outside the cage,
- (b) Antenna inside the cage,
- (c) Antenna inside the cage with ground probe connected to the cage.

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