CALIFORNIA INSTITUTE OF TECHNOLOGY
PHYSICS MATHEMATICS AND ASTRONOMY DIVISION

Sophomore Physics Laboratory (PH005/105)

Analog Electronics
Active Filters

(Revision October 2011)

Chapter 6

Active Filters

Introduction

An electronic circuit that modifies the frequency spectrum of an arbitrary signal is called filter. A filter that modifies the spectrum producing amplification is said to be an active filter. Vis-à-vis its definition, it is convenient to study the filter characteristics in terms of the frequency response of its associated two port network

$$H(\omega) = \frac{V_o(\omega)}{V_i(\omega)} \, ,$$

where V_i and V_o are respectively the input voltage and the output voltage of the network, and ω the angular frequency. Depending on the design, active filters have some important advantages:

- they can provide gain,
- they provide isolation because of the typical impedance characteristics of amplifiers,
- they can be cascaded because of the typical impedance characteristics of amplifiers,
- they can avoid the use of inductors greatly simplifying the design of the filters.

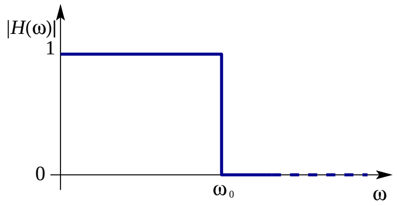
Here some disadvantages:

- they are limited by the amplifiers' band-with and noise,
- they need power supplies,
- they dissipate more heat.

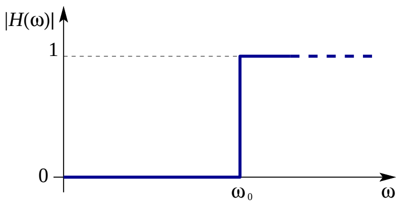
Let's make some simple definitions useful to classify different types of filters.

6.1 Classification of Ideal Filters

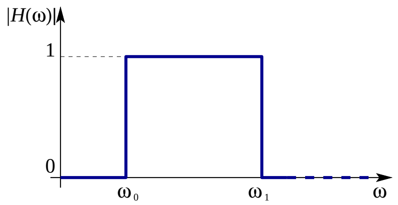
Based on their magnitude response $|H(\omega)|$, Some basic ideal filters can be classified as follows:



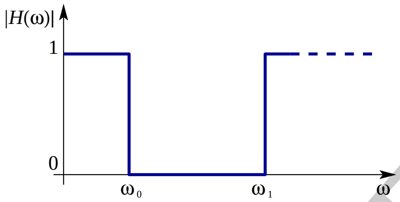
Low-pass



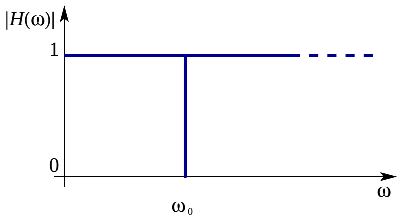
High-pass



Band-pass



Stop-band/band-reject



Notch

Practical filters approximate more or less the ideal definitions.

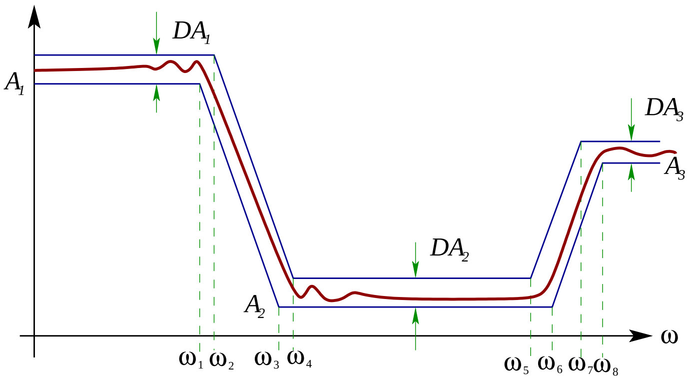


Figure 6.1: Graphical definition of the filter performance specifications, and hypothetical filter response (red curve) that satisfy the specification.

Usually, the filter requirements are specified defining the band frequencies with their gains (attenuation or amplification) gain ripples, and slope transitions in terms of power of the frequency. Figure 6.1 shows a quite general graphical definition of the design parameters of a filter with an hypothetical design. For a complete specification one should also define the requirement for phase response.

6.2 Filters as Rational Functions

Let's consider filters whose transfer function can be expressed as rational function or standard form

$$H(\omega) = \frac{\alpha_0 + \alpha_1 j\omega + \alpha_2 (j\omega)^2 + \ldots + \alpha_N (j\omega)^N}{\beta_0 + \beta_1 j\omega + \beta_2 (j\omega)^2 + \ldots + \beta_M (j\omega)^M} .$$

For the filter not to diverge

$$M \geq N \qquad \Rightarrow \qquad |H(\omega)| < \infty \text{ for any value of } \omega$$

Writing the transfer function as a polynomial factorization we obtain

$$H(\omega) = k \frac{(\omega - z_1)^{n_1} (\omega - z_2)^{n_2} \ldots (\omega - z_N)^{n_N}}{(\omega - p_1)^{m_1} (\omega - p_2)^{m_2} \ldots (\omega - p_M)^{m_M}}$$

Denominator roots $p_1, p_2, \ldots, p_n$ are called poles, and numerator roots $z_1, z_2, \ldots, z_m$ are called zeros. The integers $n_1, n_{2,} \ldots, n_N$, and $m_1, m_{2,} \ldots, m_N$ are therefore the multiplicity of poles and zeros.

Poles and zeros values determine the shape of the filter, and one could say that poles provides attenuation and zeros amplification.

The transition from transmission to attenuation and vice versa in the filter magnitude $|H(\omega)|$ is characterized by an asymptote slope which determine the so called filter order.

For example, considering the RC low pass filter with $\omega_0 = 1/RC$, we have one pole $p_1 = j\omega_0$

$$H(\omega) = \frac{\omega_0}{\omega_0 + j\omega} \Rightarrow \text{ first oder low pass filter with cut-off freq. } \omega_0$$

For example, considering the RC high pass filter with $\omega_0 = 1/RC$, we have one pole $p_1 = j\omega_0$ and one zero $z_1 = 0$

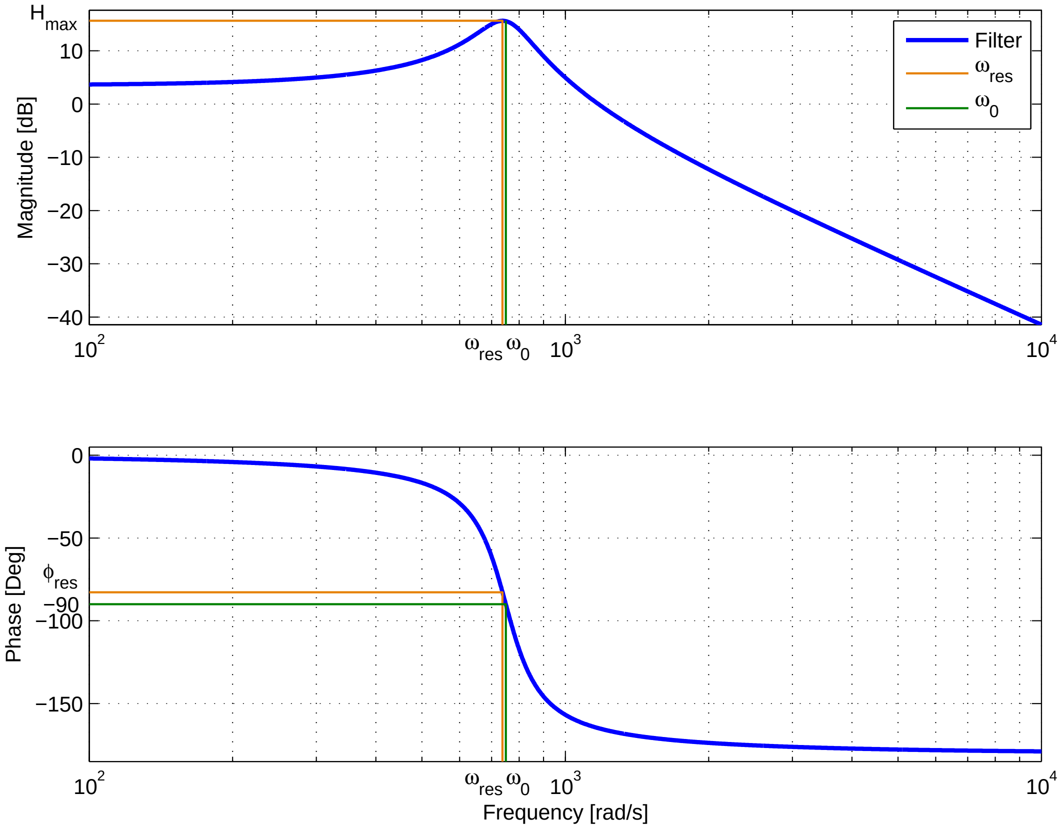
$$H(\omega) = \frac{\omega}{\omega_0 + j\omega} \Rightarrow \text{ first oder high pass filter with cut-off freq. } \omega_0$$

We will analyze into more detail filters with the following transfer function

$$H(\omega) = H_0 \frac{\omega^2 + j\omega a_1 + a_0}{-\omega^2 + j\omega \frac{\omega_0}{Q} + \omega_0^2}$$

6.2.1 Second Order Low-Pass Filter

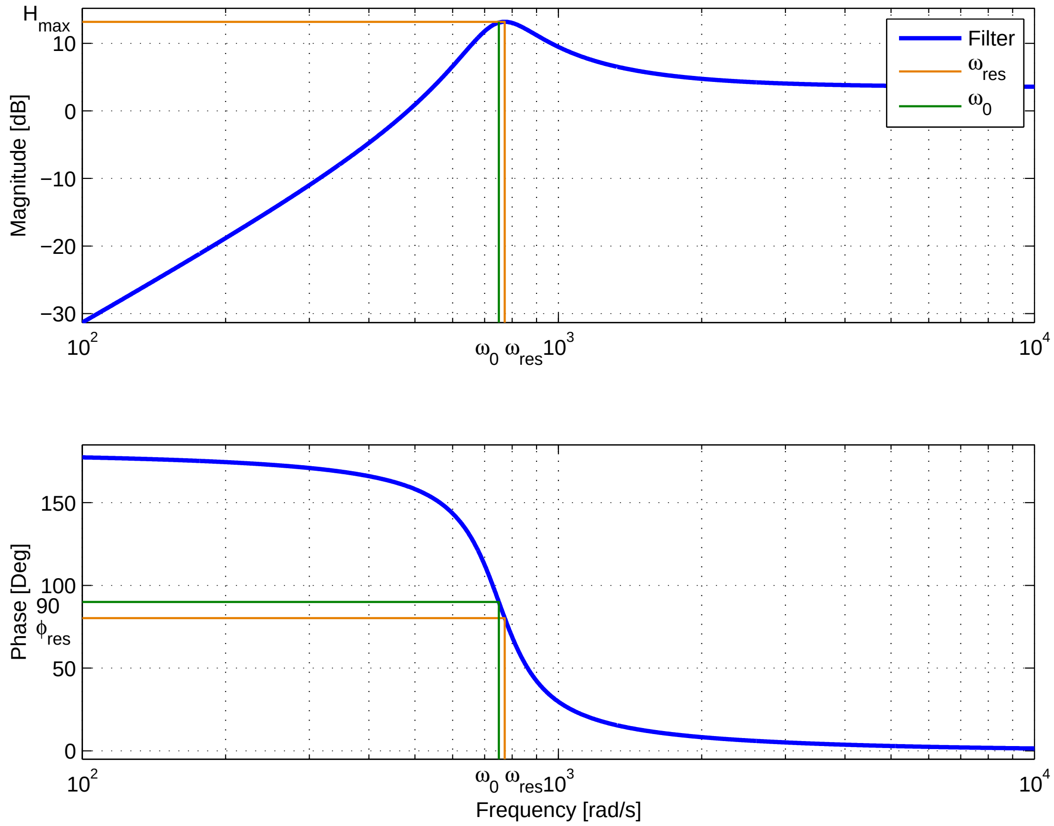
Figure below shows the second order low pass filter bode plot, with resonant frequency ω_{res}, and characteristic frequency ω_0



The second order low-pass filter written in standard form

Transfer Function	Resonance	Maximum	DC Gain	High Freq. Gain
$H_0 \dfrac{\omega_0^2}{-\omega^2 + j\omega\dfrac{\omega_0}{Q} + \omega_0^2}$	$\omega_0\sqrt{1 - \dfrac{1}{2Q^2}}$	$H_0 \dfrac{Q}{\sqrt{1 - \dfrac{1}{4Q^2}}}$	H_0	0

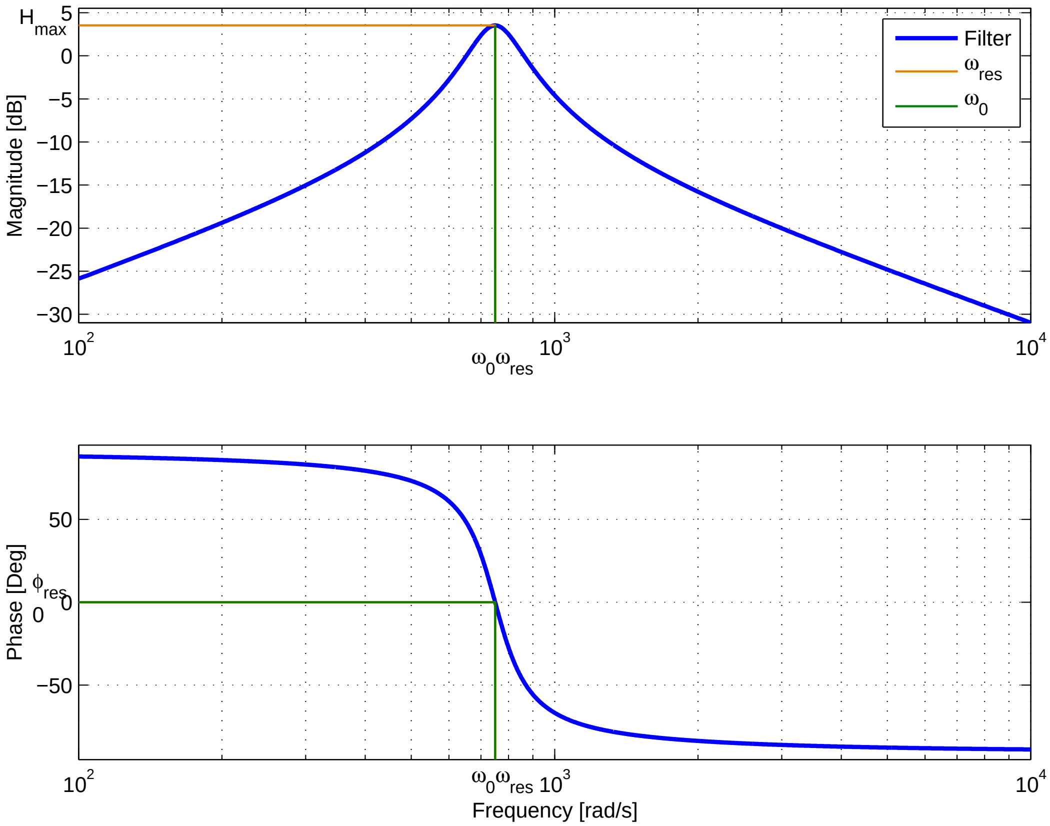
6.2.2 Second Order High-Pass Filter



The second order high-pass filter written in standard form

Transfer Function	Resonance	Maximum	DC Gain	High Freq. Gain
$H_0 \dfrac{\omega^2}{-\omega^2 + j\omega\dfrac{\omega_0}{Q} + \omega_0^2}$	$\dfrac{\omega_0}{\sqrt{1 - \dfrac{1}{2Q^2}}}$	$H_0 \dfrac{Q}{\sqrt{1 - \dfrac{1}{4Q}}}$	0	H_0

6.2.3 Band-Pass Filter



The band-pass filter written in standard form is

Transfer Function	Resonance	Maximum	DC Gain	High Freq. Gain
$H_0 \dfrac{j\omega \dfrac{\omega_0}{Q}}{-\omega^2 + j\omega \dfrac{\omega_0}{Q} + \omega_0^2}$	ω_0	H_0	0	0

For example, depending on the output we consider, the already studied LCR series circuit is a low-pass, a band-pass, or a high-pass filter with the transfer function described above. When we will study difference filters topologies we will reduce their transfer function into one of the standard form above.

6.3 Common Circuit Filters Topologies

This is a brief and not exhaustive at all list of filter topologies that use resistors, capacitors, and operational amplifiers to implement the filters types described above:

- Infinite gain, multiple feedback (IGMF). The name derives from the infinite gain of the active ideal element and double feedback topology (more that one feedback mesh).
- Generalized Sallen-Key (GSK)
- State Variable (SV)
- Switched Capacitor Filters (SC)
- Filter based on Gyrators, which implements inductors with capacitors and active components.

Cascading these implementation allows to increase the filter order.

6.4 Infinite Gain Multiple Feedback Configuration (IGMF)

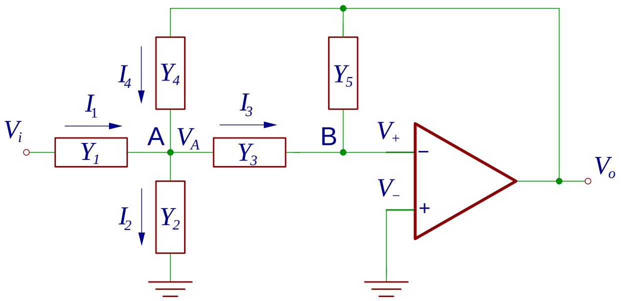


Figure 6.2: Infinite Gain Multiple Feedback Filter.

Let's consider the circuit in Figure 6.2 with generic admittances Y_1, Y_2, Y_3, Y_4, and Y_5. Applying the KCL to node A and considering the circuit virtual ground $(V_- = 0)$, we have

$$V_A Y_3 + (V_o - V_A)\,Y_4 + (V_i - V_A)\,Y_1 + V_A Y_2 = 0\,. \tag{6.1}$$

Again, applying KCL to node B and for the virtual ground we have

$$V_o Y_5 + V_A Y_3 = 0 \quad \Rightarrow \quad V_A = -\frac{Y_5}{Y_3} V_o$$

Replacing the last expression into equation (6.1) and after some algebra we obtain the generic transfer function for the circuit

$$\frac{V_o}{V_i} = -\frac{Y_1 Y_3}{Y_5\,(Y_1 + Y_2 + Y_3 + Y_4) + Y_3 Y_4}\,.$$

Choosing the proper type of admittances we can construct different types of active filters, low-pass band-pass, and high-pass. It is worthwhile noticing that IGMF configuration allows to implement low-pass, band-pass, and high-pass filter with capacitors, resistor and no inductors. This simplifies considerably the design of the filters.

6.4.1 Low-pass Filter

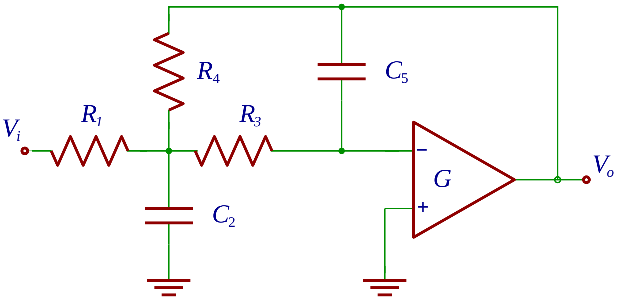


Figure 6.3: Low-pass filter configuration of the infinite gain multiple feedback filter.

A possible choice to implement a low-pass filter as shown in Figure 6.3 is

$$Y_1 = \frac{1}{R_1}, \qquad Y_2 = j\omega C_2, \qquad Y_3 = \frac{1}{R_3},$$

$$Y_4 = \frac{1}{R_4}, \qquad Y_5 = j\omega C_5,$$

and the transfer function of the circuit becomes

$$\frac{V_o}{V_i} = -\frac{{}^1\!/_{R_1 R_3}}{j\omega C_5 \left({}^1\!/_{R_1} + j\omega C_2 + {}^1\!/_{R_3} + {}^1\!/_{R_4}\right) + {}^1\!/_{R_3 R_4}}.$$

Rearranging the expression to obtain a rational fraction in ω we finally obtain

$$\frac{V_o}{V_i} = \frac{-\dfrac{1}{R_1 R_3 C_2 C_5}}{-\omega^2 + j\omega\dfrac{1}{C_2}\left({}^1\!/_{R_1} + {}^1\!/_{R_3} + {}^1\!/_{R_4}\right) + \dfrac{1}{R_3 R_4 C_2 C_5}}.$$

Comparing the denominator of the previous equation with the denominator of the transfer function in section 6.2.1 we find that the frequency ω_0, the quality factor Q, and the DC gain H_0 are respectively

$$\omega_0 = \sqrt{\frac{1}{R_3 R_4 C_2 C_5}}, \qquad Q = \omega_0 \frac{C_2}{\left({}^1\!/_{R_1} + {}^1\!/_{R_3} + {}^1\!/_{R_4}\right)}, \qquad H_0 = -\frac{R_4}{R_1}.$$

6.4.2 High-pass Filter

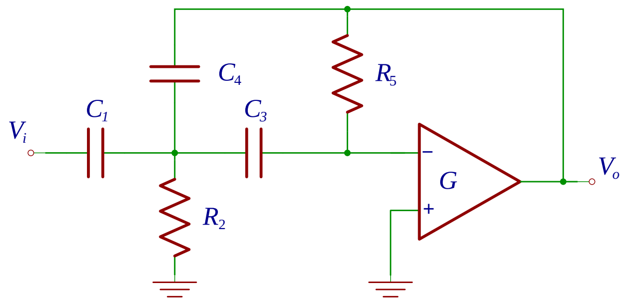


Figure 6.4: High-pass filter configuration of the infinite gain multiple feedback filter.

A possible choice to implement a high-pass filter as shown in Figure 6.4is

$$Y_1 = j\omega C_1, \qquad Y_2 = \frac{1}{R_2}, \qquad Y_3 = j\omega C_3,$$

$$Y_4 = j\omega C_4, \qquad Y_5 = \frac{1}{R_5},$$

and the transfer function of the circuit becomes

$$\frac{V_o}{V_i} = -\frac{j\omega C_1 j\omega C_3}{1/R_5\left(j\omega C_1 + 1/R_2 + j\omega C_3 + j\omega C_4\right) + j\omega C_3 j\omega C_4}.$$

Rearranging the expression to obtain a rational fraction in ω we obtain

$$\frac{V_o}{V_i} = \frac{-\omega^2\left(-C_1/C_4\right)}{-\omega^2 + j\omega\left(C_1 + C_3 + C_4\right)\frac{1}{R_5C_3C_4} + \frac{1}{R_2R_5C_3C_4}}.$$

Comparing the denominator of the previous equation with the denominator of the transfer function in section 6.2.2 we find that the frequency ω_0, the quality factor Q, High frequency gain $H_\infty$are respectively

$$\omega_0 = \sqrt{\frac{1}{R_2R_5C_3C_4}}, \qquad Q = \omega_0\frac{R_5C_3C_4}{\left(C_1 + C_3 + C_4\right)}, \qquad H_\infty = -\frac{C_1}{C_4}.$$

6.4.3 Band-pass Filter

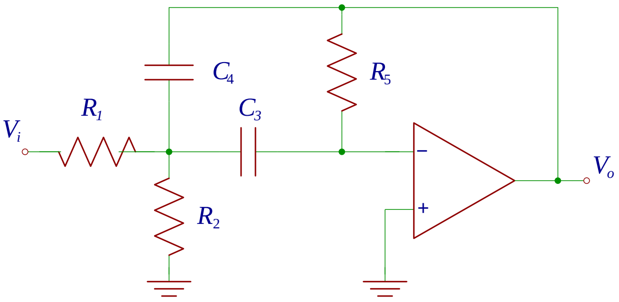


Figure 6.5: Band-pass filter configuration of the infinite gain multiple feedback filter.

A possible choice to implement a Band-pass filter is shown in Figure 6.5. The admittances are

$$Y_1 = \frac{1}{R_1}, \qquad Y_2 = \frac{1}{R_2}, \qquad Y_3 = j\omega C_3,$$

$$Y_4 = j\omega C_4, \qquad Y_5 = \frac{1}{R_5},$$

and the transfer function of the circuit becomes

$$\frac{V_o}{V_i} = -\frac{j\omega C_3/R_1}{{}^1\!/_{R_5}\left({}^1\!/_{R_1} + {}^1\!/_{R_2} + j\omega C_3 + j\omega C_4\right) + j\omega C_3 j\omega C_4}.$$

Rearranging the expression to get a rational fraction in ω we finally obtain

$$\frac{V_o}{V_i} = -C_1\frac{R_5}{R_1}\frac{j\omega\left(\frac{C_3+C_4}{R_5C_3C_4}\right)}{-\omega^2 + j\omega\frac{C_3+C_4}{C_3C_4R_5} + \frac{R_1+R_2}{R_1R_2R_5C_3C_4}}.$$

Comparing the denominator of the previous equation with the denominator of the transfer function in section 6.2.3 we find that the resonance frequency, and the quality factor are respectively

$$\omega_0 = \sqrt{\frac{R_1+R_2}{R_1R_2R_5C_3C_4}}, \qquad Q = \omega_0^2\frac{R_5C_3C_4}{C_3+C_4}, \qquad H_0 = -C_1\frac{R_5}{R_1}$$

6.5 Generalized Sallen-Key Filter Topology (GSK)

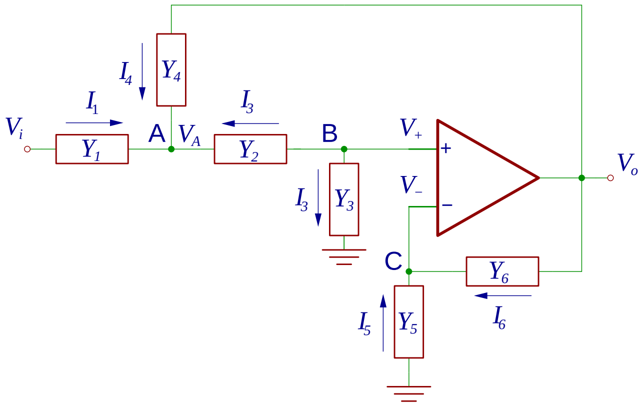


Figure 6.6: Generalized Sallen-Key Topology.

Let's consider the circuit in Figure 6.6 with generic admittances Y_1, Y_2, Y_3, Y_4, Y_5, and Y_6. Applying the KCL to node A , we have

$$(V_i - V_A)\,Y_1 + (V_o - V_A)\,Y_4 + (V_+ - V_A)\,Y_2 = 0\,. \tag{6.2}$$

Applying KCL to node B

$$(V_+ - V_A)\,Y_2 + V_+Y_3 = 0 \quad \Rightarrow \quad V_A = \frac{Y_2 + Y_3}{Y_2}V_+\,.$$

Applying KCL to node C

$$(V_0 - V_-)\,Y_6 - V_-Y_5 = 0 \quad \Rightarrow \quad V_- = V_+ = \frac{Y_6}{Y_6 + Y_5}V_0\,.$$

Replacing the expression found for V_A, and V_+ into equation (6.2) and after quite some boring algebra, we obtain

$$\frac{V_o}{V_i} = \left(1 + \frac{Y_5}{Y_6}\right)\frac{Y_1Y_2Y_6}{Y_1Y_6\,(Y_2 + Y_3) + Y_3Y_6\,(Y_2 + Y_4) - Y_2Y_4Y_5}\,.$$

Let's analyze some admittances' configuration of the this filter topology.

6.5.1 GSK Second Order Low-pass Filter

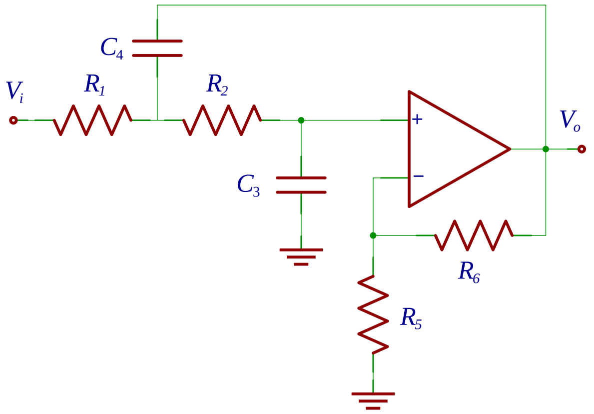


Figure 6.7: Low-pass filter configuration of the Generalized Sallen-Key filter.

A possible choice to implement a low-pass filter as shown in Figure 6.7 is

$$Y_1 = \frac{1}{R_1}, \qquad Y_2 = \frac{1}{R_2}, \qquad Y_3 = j\omega C_3,$$

$$Y_4 = j\omega C_4, \qquad Y_5 = \frac{1}{R_5}, \qquad Y_6 = \frac{1}{R_6},$$

and the transfer function of the circuit becomes

$$\frac{V_o}{V_i} = \left(1 + \frac{R_6}{R_5}\right) \frac{\frac{1}{R_1 R_2 C_3 C_4}}{-\omega^2 + j\omega\left(\frac{1}{R_1 C_4} + \frac{1}{R_2 C_4} - \frac{1}{R_2 C_3}\frac{R_6}{R_5}\right) + \frac{1}{R_1 R_2 C_3 C_4}}.$$

Comparing the denominator of the previous equation with the denominator of the transfer function in section 6.2.1we find that the frequency square ω_0^2, the quality factor Q, and the DC gain H_0 are respectively

$$\omega_0^2 = \frac{1}{R_1 C_4 R_2 C_3}, \qquad Q = \omega_0 \frac{R_1 R_2 R_5 C_3 C_4}{R_5 \left(R_1 + R_2\right) C_3 - R_1 R_6 C_4}, \qquad H_0 = \left(1 + \frac{R_6}{R_5}\right).$$

6.5.2 Simple Case

If $R_1 = R_2 = R$, $C_3 = C_4 = C$, and $R_5 = R_6 = 0$, then

$$\frac{V_o}{V_i} = \frac{\omega_0^2}{-\omega^2 + j\omega\omega_0 + \omega_0^2}, \qquad \omega_0^2 = \frac{1}{R^2C^2}, \qquad , Q = 1,$$

which is the transfer function of a second order low-pass filter with low quality factor.

6.5.3 GSK Second Order High-pass Filter

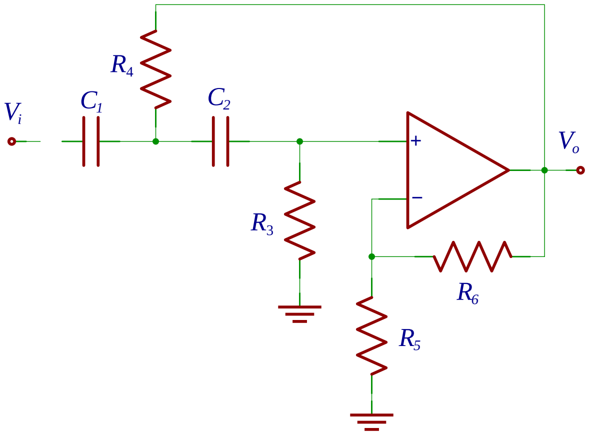


Figure 6.8: High-pass filter configuration of the Generalized Sallen-Key filter.

To implement a low-pass filter as shown in Figure 6.8 one needs to choose the admittances as follows

$$Y_1 = j\omega C_1, \qquad Y_2 = j\omega C_2, \qquad Y_3 = \frac{1}{R_3},$$

$$Y_4 = \frac{1}{R_4}, \qquad Y_5 = \frac{1}{R_5}, \qquad Y_6 = \frac{1}{R_6},$$

and the transfer function of the circuit becomes

$$\frac{V_o}{V_i} = \left(1+\frac{R_6}{R_5}\right)\frac{-\omega^2}{-\omega^2 + j\omega\left(\frac{1}{R_3C_2}+\frac{1}{R_3C_1}-\frac{1}{R_4C_1}\frac{R_6}{R_5}\right)+\frac{1}{R_3R_4C_1C_2}}\,.$$

Comparing the denominator of the previous equation with the denominator of the transfer function in section 6.2.2 we find that the frequency square ω_0^2, the quality factor Q, and the DC gain H_0 are respectively

$$\omega_0^2 = \frac{1}{R_3C_1R_4C_2}, \qquad Q = \omega_0\frac{R_3R_4R_5C_1C_2}{R_5\left(C_1+C_2\right)R_3 - C_1R_6R_4}, \qquad H_0 = \left(1+\frac{R_6}{R_5}\right).$$

6.5.4 Simple Case

If $R_1 = R_2 = R$, $C_3 = C_4 = C$, and $R_5 = R_6 = 0$, then

$$\frac{V_o}{V_i} = \frac{-\omega^2}{-\omega^2 + j\omega\omega_0 + \omega_0^2}, \qquad \omega_0^2 = \frac{1}{R^2C^2}, \qquad , Q = 1\,,$$

which is the transfer function of a second order high-pass filter with low quality factor.

6.5.5 GSK Band-pass Filter

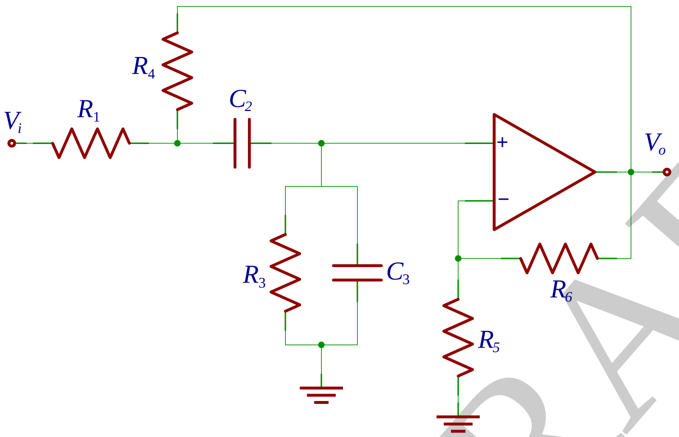


Figure 6.9: Band-pass filter configuration of the Generalized Sallen-Key filter.

To implement a band-pass filter as shown in Figure 6.9 one needs to choose the admittances as follows

$$Y_1 = \frac{1}{R_1}, \qquad Y_2 = j\omega C_2, \qquad Y_3 = \frac{1}{R_3} + j\omega C_3,$$

$$Y_4 = \frac{1}{R_4}, \qquad Y_5 = \frac{1}{R_5}, \qquad Y_6 = \frac{1}{R_6},$$

and the transfer function of the circuit becomes

$$\frac{V_o}{V_i} = \left(1 + \frac{R_6}{R_5}\right) \frac{\frac{j\omega}{R_1 C_3}}{-\omega^2 + j\omega\left(\frac{C_2 + C_3}{C_2 C_3 R_1} + \frac{1}{C_3 R_3} + \frac{1}{C_2 R_4} - \frac{1}{C_3 R_4}\frac{R_6}{R_5}\right) + \frac{R_1 + R_4}{C_2 C_3 R_1 R_3 R_4}}.$$

Comparing the denominator of the previous equation with the denominator of the transfer function in section 6.2.3 we find that the frequency square ω_0^2, the quality factor Q, and the DC gain H_0 are respectively

$$\omega_0^2 = \frac{R_1 + R_4}{C_2 C_3 R_1 R_3 R_4}, \qquad Q = \omega_0 \frac{C_2 C_3 R_1 R_3 R_4 R_5}{(C_2 + C_3) R_3 R_4 R_5 + C_2 R_1 R_4 R_5 + C_3 R_1 R_3 R_5 - C_2 R_1 R_3 R_6}, \qquad H_0 =$$

6.5.6 Simple Case

If $R_1 = R_3 = R_4 = R$, $C_2 = C_3 = C$, and $R_5 = R_6 = 0$, then

$$\frac{V_o}{V_i} = \frac{j\omega\frac{\omega_0}{Q}}{-\omega^2 + j\omega\frac{\omega_0}{Q} + \omega_0^2}, \qquad \omega_0^2 = \frac{2}{R^2C^2}, \qquad , Q = \frac{\sqrt{2}}{3},$$

which is the transfer function of a second order high-pass filter with low quality factor.

6.6 State VariableFilter Topology (SV)

TBD

6.7 Practical Considerations

6.7.1 Component Values

How do we select the values of capacitance and resistance? Here are some considerations that should help the filter design:

- reducing the resistance values reduces the thermal noise and therefore the filter noise,
- reducing resistance values minimizes the op-amp voltage offsets,
- increasing the resistance reduce the current load on the op-amps,
- increasing the resistances usually allows to decrease the capacitance and therefore it make easier to find capacitors because of the small capacitance values needed,
- reducing the capacitance minimizes the capacitance fluctuations due to temperature,
- increasing the capacitance allows to reduce resistance values and therefore the thermal noise.

As we can clearly see, some of the consideration cannot be used at the same time. Based on the design requirements one can decide which of the consideration above are more important to finally meet the design requirements.

Rules of Thumb

Particularly critical design often overrule these following rules:

- Capacitor with capacitance less of ~100 pF should be avoided,
- Try to use resistor with resistance between few kilo-ohms to few hundreds of kilo-ohms.

6.7.2 Components technology

Capacitors

The use of low loss dielectric is very important to obtain good results. If possible one should use plastic film capacitors or C0G/NPO ceramic capacitors, 1% tolerance for temperature stability.

Resistor

Low thermal noise resistors such as metal film resistors 1% tolerance for temperature stability should be used.