Ph 3: Supplemental Notes on the Forced Oscillation of the Inverted Pendulum

1 Introduction

A few changes have been made to the Inverted Pendulum apparatus which make certain sections of the lab manual only partially applicable. In particular, sections 4.5 and 4.8.2 of the manual contain information which will not be used in the actual lab. This document attempts to provide the information required for the new apparatus but is meant only as a supplement to the lab manual. As such, many definitions are not restated here.

2 Theory

The inverted pendulum (IP) apparatus we use in lab is a very good one in the sense that there is little energy loss during its operation. Consequently, the resonance peak found when studying the behavior of the apparatus under forced oscillations is quite narrow. While this allows for a precise definition of the resonance frequency, it also makes the experimental data difficult to fit; data following a high curvature curve is harder to fit than 'low curvature' data.

To make the data analysis task easier, we have artificially damped the IP by attaching a permanent magnet to the top of the pendulum such that it oscillates directly beneath a metal plate. As the magnet moves back and forth, the Lorentz force sets up currents of electrons moving within the plate. These electrons are subject to the resistivity of the metal and thus introduce a new source of friction into our system. This new energy dissipation mechanism causes the resonance peak to broaden and allows for easier data analysis.

The differential equation describing this system is

$$\ddot{x} + \gamma\dot{x} + \left(\omega_0^2 + \cancel{i\phi\omega_2^2}\right)x = 0;\ \ x = \theta l \tag{1}$$

where we have simplified the equation by assuming the energy loss due to the 'artificial' source of friction, γ, is much larger than the energy loss inherent in the flex joint, ϕ.

Displacing the system by some x_0 turns this homogeneous equation into an inhomogeneous one via the following transformation.

$$x \to x - x_0 \Rightarrow$$
$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \omega_0^2 x_0^2 \tag{2}$$

For a sinusoidal exitation, $x_0(t) = X_0\Re\left[e^{i\omega t}\right]$, the solution to equation 2 is of the form $x(t) = \Re\left[Xe^{i\omega t}\right]$ where X is some *complex* amplitude. When we substitute these equations into equation 2, we find the following.

$$H(\omega) \equiv \frac{X}{X_0} = \frac{\omega_0^2}{\omega_0^2 - \omega^2 + i\omega\gamma}$$

Defining Q, the quality factor, by $\gamma \equiv \frac{\omega_0}{Q}$, we can rewrite the transfer function as follows.

$$H(\omega) = \frac{\omega_0^2}{\omega_0^2 - \omega^2 + i\dfrac{\omega\omega_0}{Q}} \tag{3}$$

In this lab, you will be asked to measure the transfer function above. Since equation 3 defines a complex function, we must measure two *real* functions to obtain complete information about the system. The canonical choices are the magnitude and phase of $H(\omega)$.

$$|H(\omega)| = \frac{\omega_0^2}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + \left(\frac{\omega\omega_0}{Q}\right)^2}} \quad (4)$$

$$\phi(\omega) \equiv \arg H(\omega) = -\arctan\left(\frac{\frac{\omega\omega_0}{Q}}{\omega_0^2 - \omega^2}\right) \quad (5)$$

For the magnitude, we can show that the maximum, ω_{max}, occurs near ω_0 and that

$$\begin{aligned} |H(\omega \ll \omega_0)| &\simeq 1 \\ |H(\omega = \omega_0)| &= Q \\ |H(\omega \gg \omega_0)| &\simeq \frac{\omega_0^2}{\omega^2} \end{aligned}$$

For the phase, we see

$$\begin{aligned} \phi(\omega \ll \omega_0) &\simeq 0 \\ \phi(\omega = \omega_0) &= -\frac{\pi}{2} \\ \phi(\omega \gg \omega_0) &\simeq -\pi \end{aligned}$$

3 Experiment

In this lab, you will use the Matlab function `IPTransmissibility` to record data for a single load mass over a range of driving frequencies. Make sure to turn the amplitude of the function generator down to zero before changing frequencies, to keep the amplitude low enough that the driving force really is sinusoidal, and to wait an appropriate amount of time after changing the frequency to allow the transient signals to die out before taking data.

You will use fit functions `FitLowPass2ndOrderMag` and `FitLowPass2ndOrderPhase` to fit the magnitude and phase data, respectively. Report on the resonance frequency, ω_{max}, and on the quality factor, Q.