CALIFORNIA INSTITUTE OF TECHNOLOGY
PHYSICS MATHEMATICS AND ASTRONOMY DIVISION

Freshman Physics Laboratory (PH003)

Analog Electronic Measurements: AC Measurement

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Contents

Chapter 10

Alternating Current Network Theory

In this chapter we will study the properties of electronic networks propagating sinusoidal currents (alternate currents/AC). In this case, voltage or current sources produce sinusoidal waves whose frequency can be ideally changed from 0 to ∞. The AC analysis of such circuits is valid once the network is at the steady state, i.e. when the transient behavior (such as that produced by closing or opening switches) is extinguished.

In general, if we have a sinusoidal signal (voltage, or current) applied to a circuit having at least one input and one output, we will expect a change in amplitude and phase at the output. The determination of these quantities for quite simple circuits can be very complex. It is indeed important to develop a convenient representation of sinusoidal signals to simplify the analysis of circuits in the AC regime.

10.1 Symbolic Representation of a Sinusoidal Signals, Phasors

A sinusoidal quantity (a sinusoidal current or voltage for example) ,

$$A(t) = A_0 \sin(\omega t + \varphi),$$

is completely characterized by the amplitude A_0, the angular frequency ω, and the initial phase φ. The phase φ corresponds to a given time shift t^* of the sinusoid ($\omega t^* = \varphi \ \Rightarrow t^* = \varphi/\omega$).

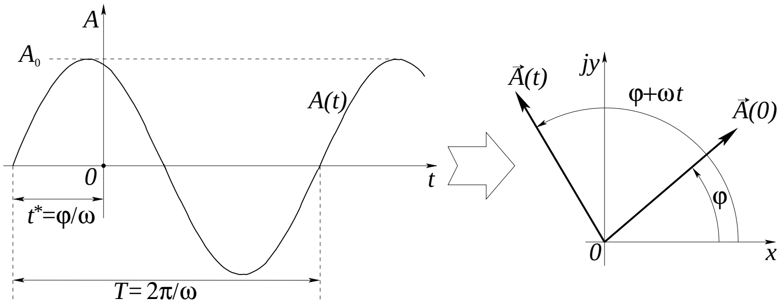

Figure 10.1: Sinusoidal quantity $A(t)$ and its phasor representation $\vec{A}$ at the initial time $t = 0$ and at time t.

We can indeed associate to $A(t)$ an applied vector $\vec{A}$ of the complex plane with modulus $|\vec{A}| = A_0 \geq 0$, rotating counter-clock wise around the origin O, with angular frequency ω, and initial angle φ (see figure 10.1). This vector is called *phasor.*

The complex representation of the phasor is indeed[1]

$$\vec{A} = A_0 e^{j(\omega t+\varphi)}, \qquad j = \sqrt{-1},$$

or

$$\vec{A} = x + jy, \qquad \begin{cases} x = A_0 \cos(\omega t + \varphi) \\ y = A_0 \sin(\omega t + \varphi) \end{cases}$$

Extracting the real and the imaginary part of the phasor, we can easily compute its amplitude A_0 and phase φ, i.e.

$$|\vec{A}| = \sqrt{\Re[\vec{A}]^2 + \Im[\vec{A}]^2} \qquad \varphi = \arg\left[\vec{A}\right] = \arctan\left(\frac{\Im[\vec{A}]}{\Re[\vec{A}]}\right), \quad (t = 0)$$

and reconstruct the real sinusoidal quantity. It is worthwhile to notice that in general, amplitude A_0 and phase φ are functions of the frequency.

The convenience of this representation will be evident, once we consider the operation of derivation and integration of a phasor.

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[1] To avoid confusion with the symbol of the electric current i, it is convenient to use the symbol j for the imaginary unit.

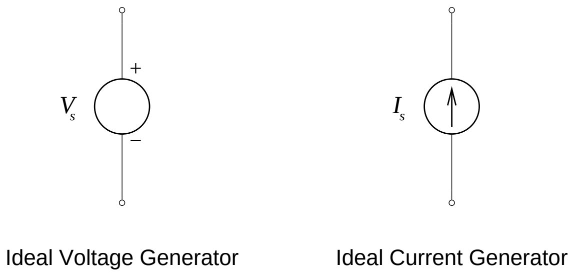


Figure 10.2: Ideal voltage and current generators symbols.

10.1.1 Derivative of a Phasor

Computing the derivative of a phasor $\vec{A}$, we get

$$\frac{d\vec{A}}{dt} = j\omega A_0 e^{j(\omega t+\varphi)} = j\omega\vec{A},$$

i.e. the derivative of a phasor is equal to the phasor times $j\omega$.

10.1.2 Integral of a Phasor

The integral of a phasor $\vec{A}$ is

$$\int_{t_0}^{t} \vec{A}dt' = \frac{1}{j\omega}\left[A_0 e^{j(\omega t+\varphi)} - A_0 e^{j(\omega t_0+\varphi)}\right] = \frac{1}{j\omega}\vec{A} + \text{const.},$$

i.e. the integral of a phasor is equal to the phasor divided by $j\omega$ plus a constant. For the AC regime we can assume the constant to be equal to zero without loss of generality.

Symbols for ideal sinusoidal voltage and current generators are shown in figure 10.2.

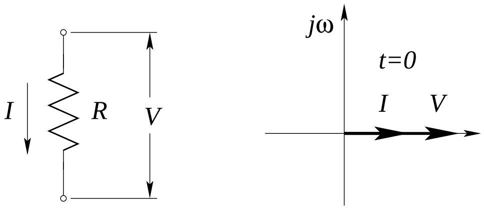
10.2 Current Voltage Equation for Passive Ideal Components with Phasors

Let's rewrite the I-V characteristic for the passive ideal components using the phasor notation. For sake of simplicity, we remove the arrow above the phasor symbol. To avoid ambiguity, we will use upper case letters to indicate phasors, and lower case letters to indicate a generic time dependent signal.

10.2.1 The Resistor

For time dependent signals, Ohm's law for a resistor with resistance R is

$$v(t) = Ri(t).$$



Introducing the phasor $I = I_0 e^{j\omega t}$ (see figure above), we get

$$v(t) = RI_0 e^{j\omega t},$$

and in the phasor notation

$$V = R\,I.$$

Note that in this case, the frequency and time dependence is implicitly contained in the phasor current I.

10.2.2 The Capacitor

The variation of the voltage difference dv across a capacitor with capacitance C, and due to the amount of charge dQ, is

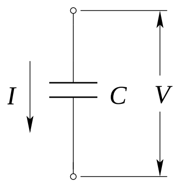
$$dv = \frac{dQ}{C}.$$

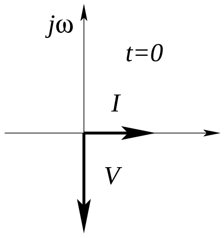
If the variation happens in a time dt, and considering that

$$i(t) = \frac{dQ}{dt},$$

we will have

$$\frac{dv(t)}{dt} = \frac{1}{C} i(t), \quad \Rightarrow \quad v(t) = \frac{1}{C}\int_0^t i(t')dt' + v(0).$$





Introducing the phasor $I = I_0 e^{j\omega t}$ (see figure above), we get

$$v(t) = \frac{1}{C}\int_0^t I_0 e^{j\omega t'} dt' + v(0),$$

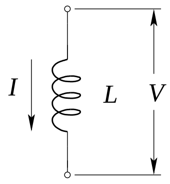
Using the phasor notation and supposing that for $t = 0$ the capacitor is discharged, we finally get

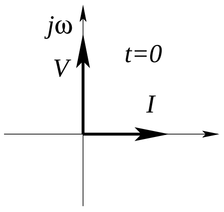
$$V = \frac{1}{j\omega C} I, \qquad , v(0) = 0.$$

10.2.3 The Inductor

The induced voltage $v(t)$ of an inductor with inductance L, is

$$v(t) = L\frac{di(t)}{dt}.$$





Introducing the phasor $I = I_0 e^{j\omega t}$ (see figure above), we get

$$v(t) = L\frac{d}{dt} I_0 e^{j\omega t},$$

and in the phasor notation

$$V = j\omega L\, I.$$

10.3 The Impedance and Admittance Concept.

Let's consider a generic circuit with a port, whose voltage difference and current are respectively the phasors $V = V_0 e^{j(\omega t+\varphi)}$, and $I = I_0 e^{j(\omega t+\psi)}$. The ratio Z between the voltage difference and the current

$$Z(\omega) = \frac{V}{I} = \frac{V_0}{I_0} e^{j(\varphi-\psi)}.$$

is said to be the *impedance of the circuit.*

The inverse

$$Y(\omega) = \frac{1}{Z(\omega)}$$

is called the *admittance of the circuit.*

For example, considering the results of the previous subsection, the impedance for a resistor, a capacitor, and an inductor are respectively

$$Z_R = R, \qquad Z_C(\omega) = \frac{1}{j\omega C}, \qquad Z_L(\omega) = j\omega L,$$

and the admittances are

$$Y_R = \frac{1}{R}, \qquad Y_C(\omega) = j\omega C, \qquad Y_L(\omega) = \frac{1}{j\omega L}.$$

In general, the impedance and the admittance of a circuit port is a complex function, which depends on the angular frequency ω. Quite often they are graphically represented by plotting their magnitude and phase .

10.3.1 Impedance in Parallel and Series

It can be easily demonstrated that the same laws for the total resistance of a series or a parallel of resistors hold for the impedance

$$Z_{tot} = Z_1 + Z_2 + ... + Z_N, \quad \text{(impedances in series)}$$

$$\frac{1}{Z_{tot}} = \frac{1}{Z_1} + \frac{1}{Z_1} + ... + \frac{1}{Z_N}, \quad \text{(impedances in parallel)}$$

10.3.2 Ohm's Law for Sinusoidal Regime

Thanks to the impedance concept, we can generalize Ohm's law and write the fundamental equation (*Ohm's law for sinusoidal regimes*)

$$V(\omega) = Z(\omega)I(\omega).$$

10.4 Two-port Network

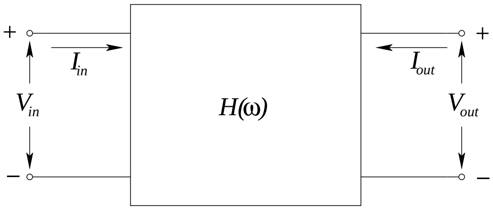


Figure 10.3: Two-port network circuit representation. The Voltage difference signs and current directions are conventional.

A linear circuit with one input and one output is called a two-port network (see figure 10.3).

To characterize the behavior of a two-port network, we can study the response of the output V_o as a function of the angular frequency ω of a sinusoidal input V_i.

In general, we can write

$$V_o(\omega) = H(\omega)V_i(\omega), \quad \text{or} \quad H(\omega) = \frac{V_o(\omega)}{V_i(\omega)},$$

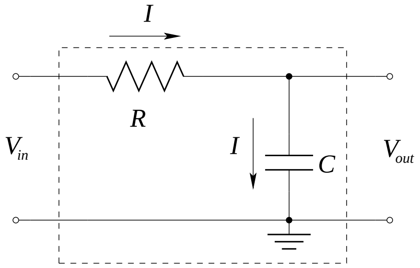


Figure 10.4: RC low-pass filter circuit.

where the complex function $H(\omega)$ is called the *transfer function of the two-port network*. The transfer function contains the information of how the amplitude and the phase of the input changes when it reaches the output. Knowing the transfer function of a two-port network, we characterize completely the circuit[2]. The definition of $H(\omega)$ suggests the way of measuring the transfer function. In fact, exciting the input with a sinusoidal wave we can measure the amplitude and the phase lead or lag respect to the input of the output signal.

To graphically represent $H(\omega)$, it is common practice to plot the magnitude $|H(\omega)|$ in a double logarithmic scale, and the phase $\arg\left[H(\omega)\right]$ in a semilogarithmic scale for the angular frequency.

It is important to notice that it is not necessary to have an ideal sinusoidal generator to make the transfer function measurement. In fact, if the input amplitude changes with the frequency the ratio between the output will not change. The same is true for the phase, i.e. if the input phase changes the difference with the output phase cannot change.

Let's study three common two port networks, the RC low-pass filter, the RC high-pass filter and the LCR series resonant circuit.

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[2] A much deeper understanding of the transfer function , requires the concept of the Fourier transform and the Laplace transform.

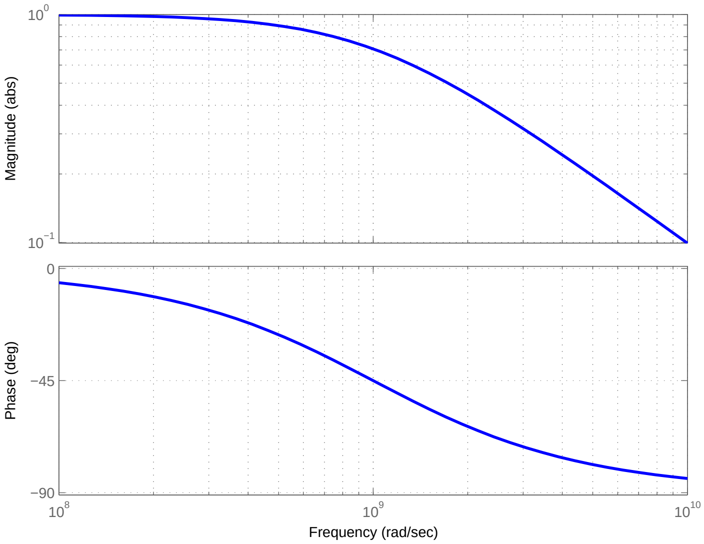

Figure 10.5: RC low-pass filter circuit transfer function.

10.4.1 The RC Low-Pass Filter

Figure 10.4 shows the *RC low-pass filter circuit*. The input and the output voltage differences are respectively[3]

$$\begin{aligned} V_{in} &= Z_{in} I = \left(R + \frac{1}{j\omega C}\right) I, \\ V_{out} &= Z_{out} I = \frac{1}{j\omega C} I, \end{aligned}$$

and the transfer function is indeed

$$H(\omega) = \frac{V_{out}}{V_{in}} = \frac{1}{1 + j\tau\omega}, \qquad \tau = RC.$$

or

$$H(\omega) = \frac{1}{1 + j\omega/\omega_0}, \qquad \omega_0 = \frac{1}{RC}.$$

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[3] V_{out} as function of V_{in} can be directly calculated using the voltage divider equation.

Computing the magnitude and phase of $H(\omega)$, we obtain

$$\begin{aligned} |H(\omega)| &= \frac{1}{\sqrt{1+\tau^2\omega^2}} \\ \arg(H(\omega)) &= -\arctan\left(\frac{\omega}{\omega_0}\right) \end{aligned}$$

Figure 10.5 shows the magnitude and phase of $H(\omega)$. The parameter τ and ω_0 are called respectively the *time constant* and the *angular cut-off frequency of the circuit*. The cut-off frequency is the frequency where the output V_{out} is attenuated by a factor $1/\sqrt{2}$.

It is worthwhile to analyze the qualitative behavior of the capacitor voltage difference V_{out} at very low frequency and at very high frequency.

For very low frequency the capacitor is an open circuit and V_{out} is essentially equal to V_{in}. for high frequency the capacitor acts like a short circuit and V_{out} goes to zero.

The capacitor produces also a delay as shown in the phase plot. At very low frequency the V_{out} follows V_{in} (they have the same phase). The output V_{out} loses phase ($\omega t = \varphi \Rightarrow t = \varphi/\omega$) when the frequency increases the frequency V_{out} starts lagging due to the negative phase ,φ, and then reaches a maximum delay due to a phase shift of $-\pi/2$.

10.4.2 The CR High-Pass Filter

Figure 10.6 shows the *CR high-pass filter circuit*. The input and the output voltage differences are respectively

$$\begin{aligned} V_{in} &= Z_{in} I = \left(R + \frac{1}{j\omega C}\right) I, \\ V_{out} &= Z_{out} I = RI, \end{aligned}$$

and indeed the transfer function is

$$H(\omega) = \frac{V_{out}}{V_{in}} = \frac{j\omega\tau}{1+j\tau\omega}, \qquad \tau = RC.$$

or

$$H(\omega) = \frac{j\omega/\omega_0}{1+j\omega/\omega_0}, \qquad \omega_0 = \frac{1}{RC}.$$

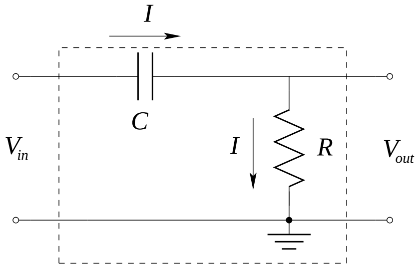


Figure 10.6: CR high pass filter circuit

Computing the magnitude and phase of $H(\omega)$, we obtain

$$\begin{aligned} |H(\omega)| &= \frac{\tau\omega}{\sqrt{1+\tau^2\omega^2}}, \\ \arg(H(\omega)) &= \arctan\left(\frac{\omega_0}{\omega}\right) \end{aligned}$$

Figure 10.7 shows the magnitude and phase of $H(\omega)$. The definitions in the previous subsection for τ, and ω_0 hold for the RC high-pass filter.

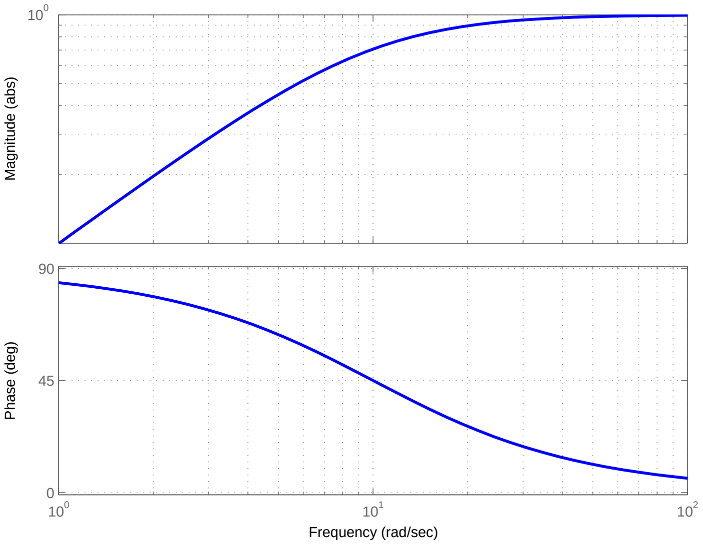


Figure 10.7: CR high pass filter circuit transfer function.

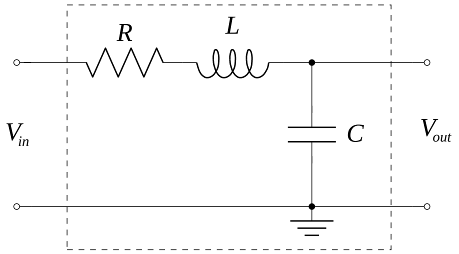


Figure 10.8: LCR series resonant circuit.

10.4.3 The LCR Series Resonant Circuit

Figure 10.8 shows the *LCR series circuit*. Considering the voltage difference across the capacitor as the circuit output, we will have

$$\begin{aligned} V_{in} &= \left(R + j\omega L + \frac{1}{j\omega C} \right) I, \\ V_{out} &= \frac{1}{j\omega C} I\,, \end{aligned}$$

and the transfer function will be

$$H_C(\omega) = \frac{1}{j\omega RC - \omega^2 LC + 1}.$$

Computing the magnitude and phase of $H(\omega)$, we obtain

$$\begin{aligned} |H_C(\omega)| &= \frac{1}{\sqrt{\left(1-\omega^2 LC\right)^2 + \left(\omega RC\right)^2}} \\ \arg\left[H_C(\omega)\right] &= \arctan\left(\frac{\omega RC}{\omega^2 LC - 1}\right) \end{aligned}$$

For sake of simplicity It is convenient to define the two following quantities

$$\omega_0^2 = \frac{1}{LC}, \qquad Q = \frac{1}{R}\sqrt{\frac{L}{C}}\,.$$

The parameter Q is called the quality factor of the circuit. Considering the previous definitions, and after some algebra $H_C(\omega)$ can be rewritten as

$$H_C(\omega) = \frac{\omega_0^2}{-\omega^2 + j\omega\frac{\omega_0}{Q} + \omega_0^2}\,. \tag{10.1}$$

The magnitude has an absolute maximum for

$$\omega_C^2 = \omega_0^2\left(1 - \frac{1}{2Q^2}\right), \qquad \text{(angular resonant frequency)}$$

and the maximum is

$$|H_C(\omega_C)| = \frac{Q}{\sqrt{1-\frac{1}{2Q^2}}}, \qquad \text{if } Q \gg 1 \quad \Rightarrow \omega_C \simeq \omega_0,\ |H_C(\omega_C)| \simeq Q$$

Far from resonance the approximate behavior of $|H_C(\omega)|$ is

$$\begin{array}{rclcl} \omega & \ll & \omega_C & \Rightarrow & |H_C(\omega)| \simeq 1 \\ \omega & \gg & \omega_C & \Rightarrow & |H_C(\omega)| \simeq \dfrac{\omega_0^2}{\omega^2} \end{array}$$

Figure 10.9 shows the magnitude and phase of $H_C(\omega)$.

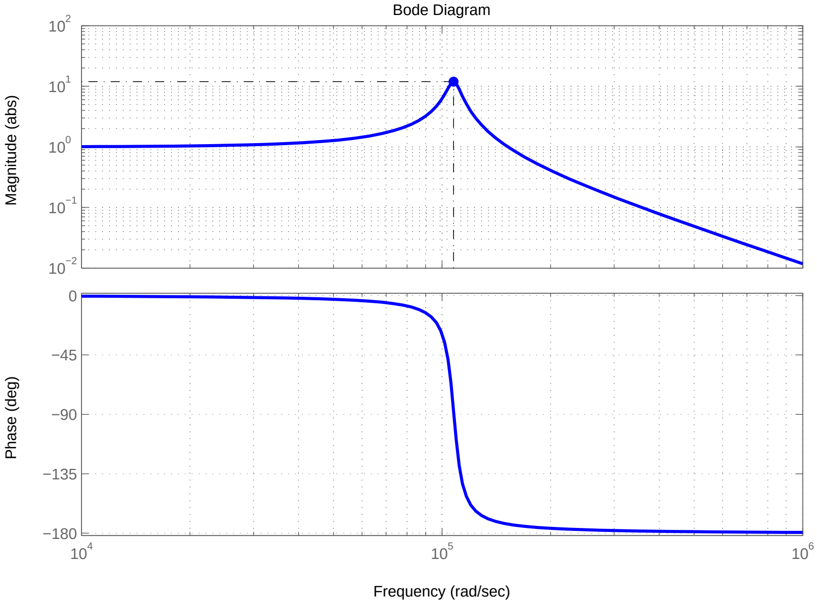


Figure 10.9: Transfer Function $H_C(\omega)$of the LCR series resonant circuit .

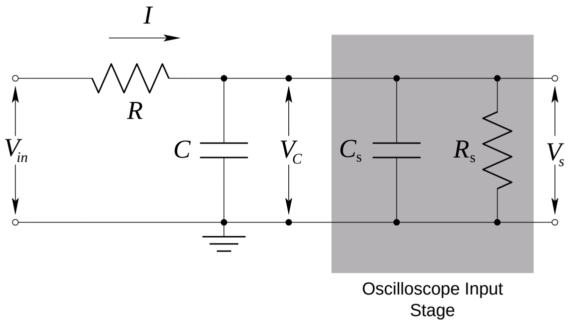
10.5 First Laboratory Week

10.5.1 Pre-laboratory Exercises

1. Calculate the total impedance of a series of a resistor with a capacitor and for a parallel of a resistor with a capacitor. Do your results confirm the statement that capacitors behave as a short circuit at high frequencies and as an open circuit at low frequencies ?

2. Calculate the magnitude of the total impedance for a series of a resistor with a capacitor having $R = 10\text{k}\Omega$, $C = 2.5\text{nF}$, and $v = 20\text{kHz}$. Calculate the same quantity for a parallel of a resistor with a capacitor having $R = 10\text{M}\Omega$, $C = 30\text{pF}$, and $v = 20\text{kHz}$.

3. The circuit shown below includes the impedance of the input channel of the CRT oscilloscope, and V_s is indeed the real voltage measured by the instrument.



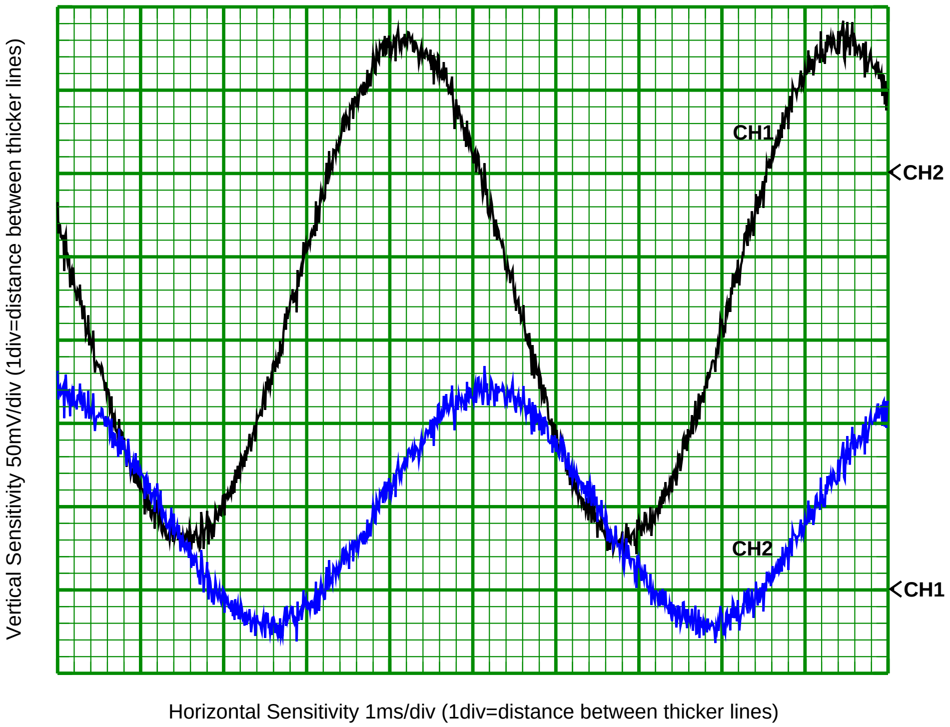
   Find the voltage $V_s(\omega)$, and the angular cut-off frequency ω_0 of the transfer function V_s/V_{in}(i.e. the value of ω for which $|V_s/V_{in}| = 1/\sqrt{2}$).
   Show that for $\omega = 0$ the $V_s(\omega)$ formula simplifies and becomes the resistive voltage divider equation.
   Demonstrate that the conditions to neglect the input impedance of the oscilloscope are the following :

$$C \gg C_s, \qquad R \ll R_s$$

4. Considering the previous circuit, calculate the value of R to obtain $V_{in} \simeq V_s$ with a fractional systematic error of 1%, if $\omega = 0\text{rad/s}$ and $R_s = 1\text{M}\Omega$.

5. Read the first two sections of the oscilloscope notes (see appendix **??**).

6. Considering the figure below (a "snapshot" of an oscilloscope display), determine the peak to peak amplitude, the DC offset, the frequency of the two sinusoidal curves, and the phase shift between the two curves (channels horizontal axis position is indicated by an arrow and the channel name on the right of the figure).



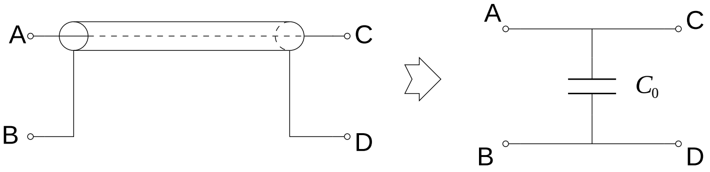
10.5.2 Procedure

Read carefully the text before starting the laboratory measurements.

Note that instead of the angular frequency ω, this procedure reports the frequency ν ($\omega = 2\pi\nu$), which is more often used in laboratory measurements.

"BNC" cables and wires terminated with "banana" connectors are available to connect circuits to the available instruments.

BNC[4] cables, a diffused type of radio frequency (RF) coaxial cable, have an intrinsic capacitance due to their geometry as shown in the figure below



They have typical linear density capacitance $\Delta C/\Delta l \sim 100\text{pF/m}$. Single wires have usually smaller capacitance than BNC cables. Unfortunately, their capacitance strongly depends on how they are positioned one respect to the others.

- Build a RC low-pass filter or a CR high-pass filter with a cut-off frequency ν_0 between 1kHz and 100kHz, using values for R and C, which makes the input impedance of the oscilloscope $Z_s$negligible compared to the impedance of your circuit.
- Measure the transfer function $H(\nu)$ of your circuit by measuring $|H(\nu)|$, and $\arg[H(\nu)]$. Then, fit the experimental data with the proper theoretical curves.
- Experimentally find $|H(\nu_0)|$ and $\arg[H(\nu_0)]$ (at the cut-off frequency ν_0), and compare them with the theoretical data.
- Build a RC low-pass filter using a capacitance C comparable with the input capacitance C_s of the oscilloscope, and check the perturbation induced by the instrument at the cut-off frequency ν_0.

10.6 Second Laboratory Week

10.6.1 Pre-laboratory Exercises

1. Considering an inductor made of a coil with large inductance ($L \sim 10\text{mH}$), resistance $R_L = 80\Omega$, wire diameter $d = 100\mu\text{m}$, and resis-

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[4] "BNC" seems to stand for Bayonet Neill Concelman (named after Amphenol engineer Carl Concelman). Other sources claim that the acronym means British Navy Connector. What is certain is that the BNC connector was developed in the late 1940's as a miniature version of the type C connector (what does the "C" stand for ?)

tivity of $\rho \simeq 16\mathrm{n}\Omega \cdot \mathrm{m}$ (copper), determine the length l of the coil wire.

2. Demonstrate that the magnitude of the LCR series transfer function $H_C(\omega)$ has a maximum for $\omega = \omega_C$ and the maximum is equal to the quality factor Q, ($Q \gg 1$).

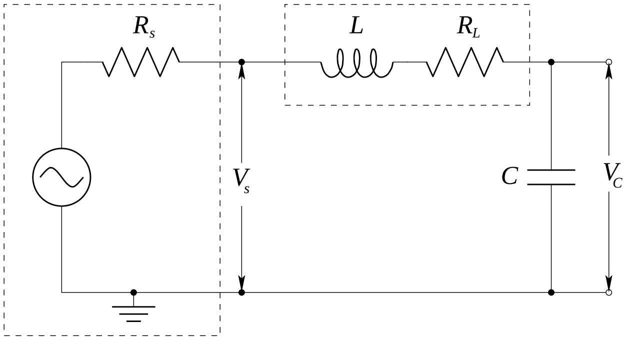
3. Supposing that R, ω_0, and Q of an LCR series circuit are known determine L, and C.

4. Compute the Magnitude and phase of equation (10.1).

5. Determine the phase of the LCR series transfer function $H_C(\omega)$ at $\omega = 0$, at the angular resonant frequency ω_C, and for $\omega \to \infty$.

6. Determine the capacitance C of a LCR series circuit necessary to have a resonant frequency $\nu_C = 20\mathrm{kHz}$ if $L = 10\mathrm{mH}$, and $R = 80\Omega$. Then, calculate the quality factor of the LCR series circuit.

10.6.2 Procedure



- Construct a LCR series resonant circuit above with a resonant frequency ν_C of about 20kHz. Note that the resistance R_s is the internal resistance of the function generator, and the resistance R_L is the resistance of the inductor. Use a mylar capacitor to make the circuit.

- Measure magnitude and phase of the transfer function

$$H_C(\nu) = \frac{V_C}{V_s}.$$

- Fitting the magnitude and phase of H_C, determine the frequency ν_0 and the quality factor Q .
- Experimentally find the resonant frequency ν_C where the maximum of $|H_C(\nu)|$ occurs and compare it with the value indirectly obtained from the fits.
- Experimentally find the resonant frequency ν_C where the proper phase shift for the resonant frequency occurs and compare it with the value obtained from the fits.
- Directly measuring the values of R, L, and C, calculate ν_0, and Q and compare with the values obtained from the fits.
- Assuming that the direct measurement of R is correct, using ν_0 and Q values obtained from the fits, estimate L and C.