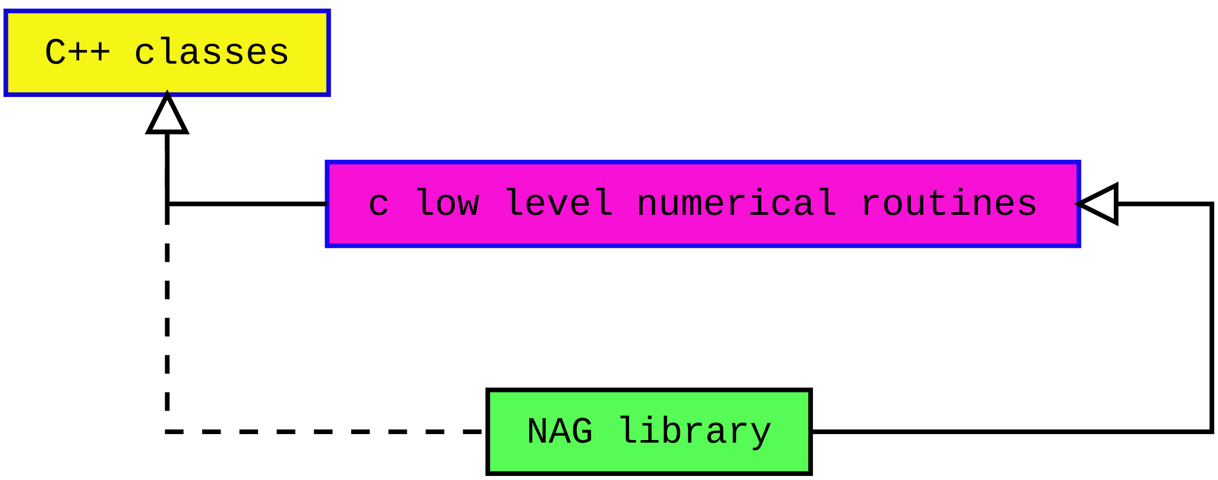
MSE- A mechanical simulation engine for the LIGO E2E model

Giancarlo Cella

Caltech,11/23/99

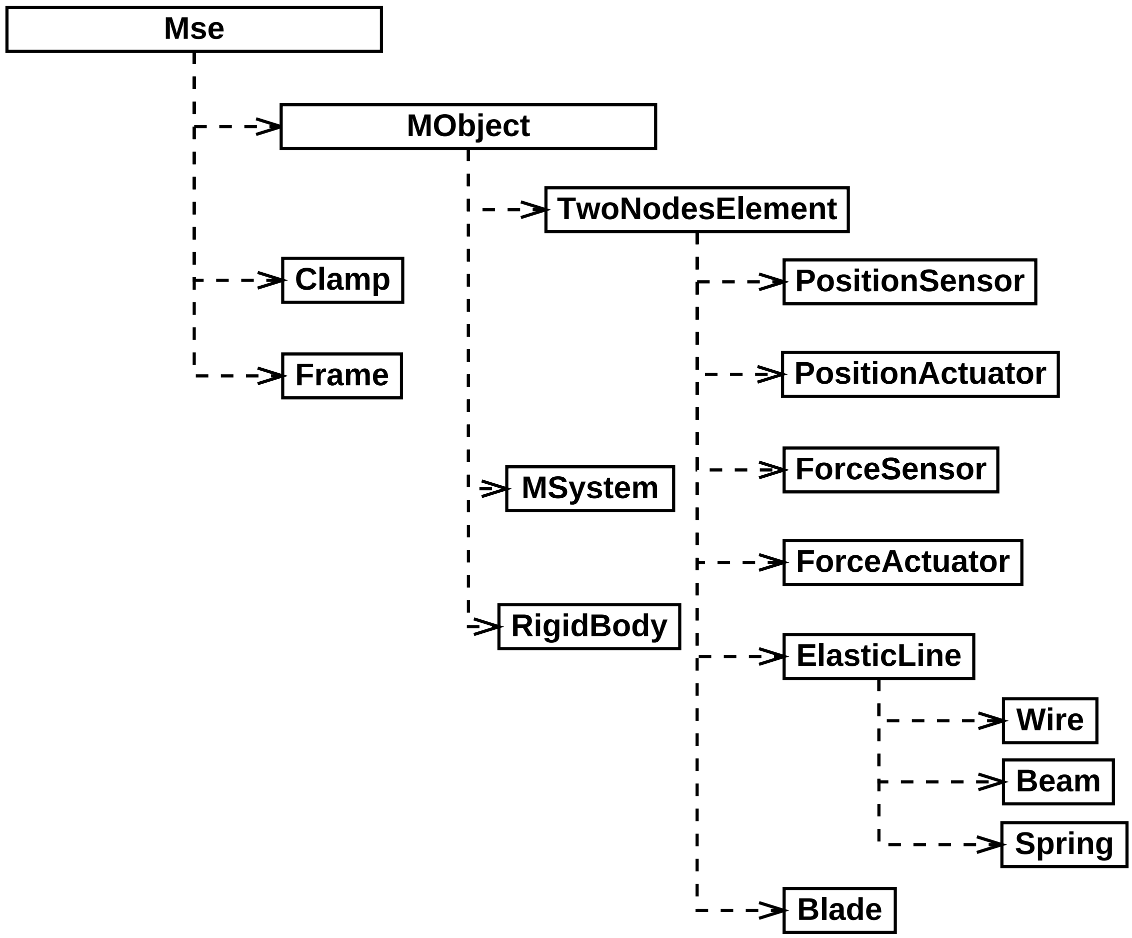
1. **General structure of the library**
2. **How to use it**
3. **Modelization techniques and problems**
4. **Future developments**



General principles

- ✔ **It is a fully tridimensional simulation. In this way it is possible to give extimates on cross couplings connected to system asymmetries.**
- ✔ **It is a modular environment. The system is partitioned in subunities, and each of them can be modeled internally in an arbitrary way.**
- ✔ **A model can be selectively refined. For example it is possible to set the number of internal modes of a given mechanical component, or to choose different representation for it.**
- ✔ **The equilibrium working point for the system is automatically calculated. There is no need to insert a lot of geometrical positioning parameters, only the connections between elements must be specified.**
- ✔ **It is (hopefully) easy to use.**

Implementation

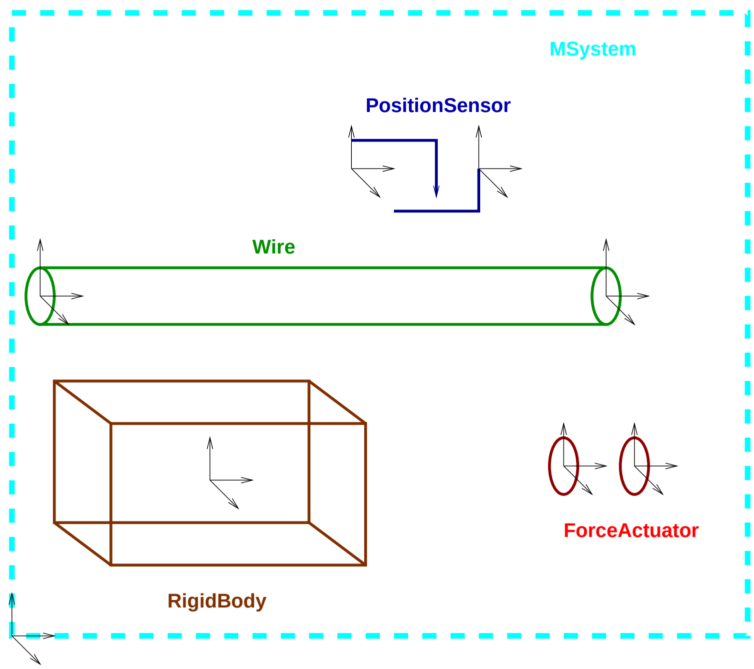


- ✔ **Frame:** is a specification of **position** and orientation of a point (6 d.o.f.)
- ✔ **Clamp:** is a rigid connection between two frames
- ✔ **MObject:** is a collection of frames with some dynamic defined among them
- ✔ **MSystem:** is an inertial frame, and also a container of MObjects

A simple example: suspended mirror

First we declare the system and the objects which compose it:

```
MSystem pendulum;
RigidBody mirror;
Wire wire1,wire2;
ForceActuator coil1,coil2,coil3,coil4;
PositionSensor sensor;
```

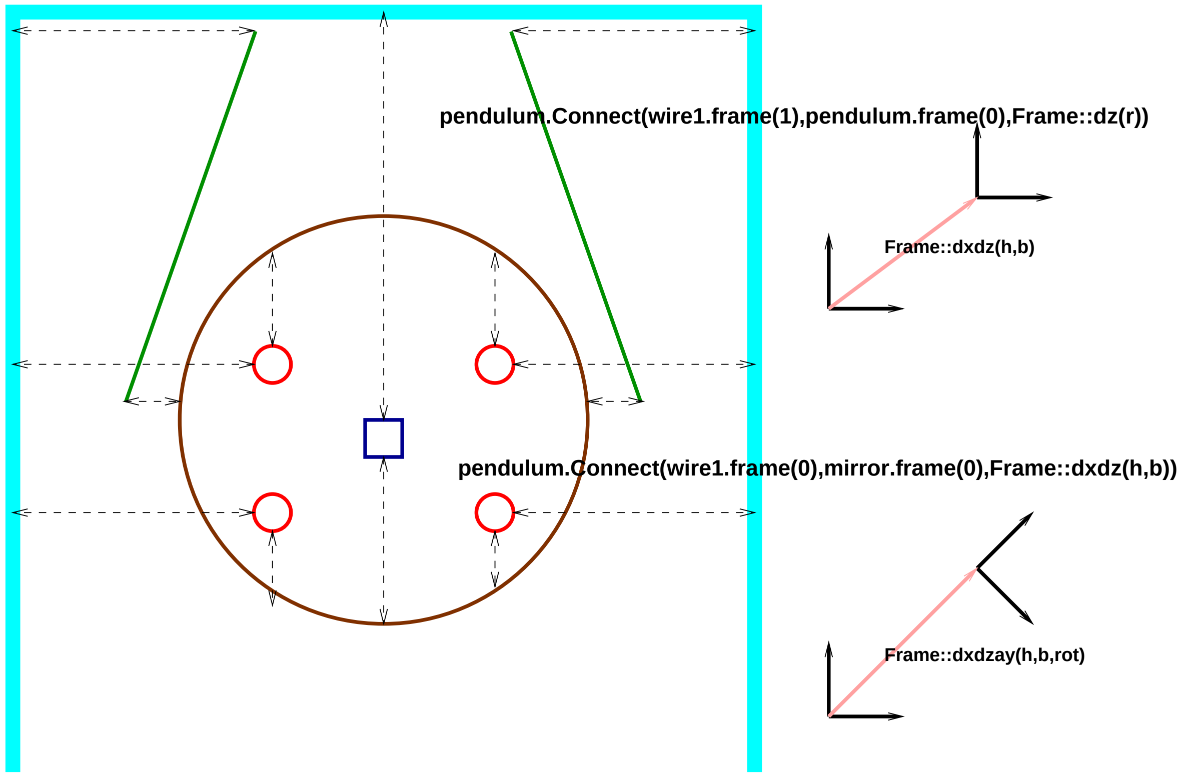


And we set the relevant parameters (mass, inertia tensor components, wire diameter etc.)

Now the system must actually be constructed. This is obtained clamping frames together.

System construction

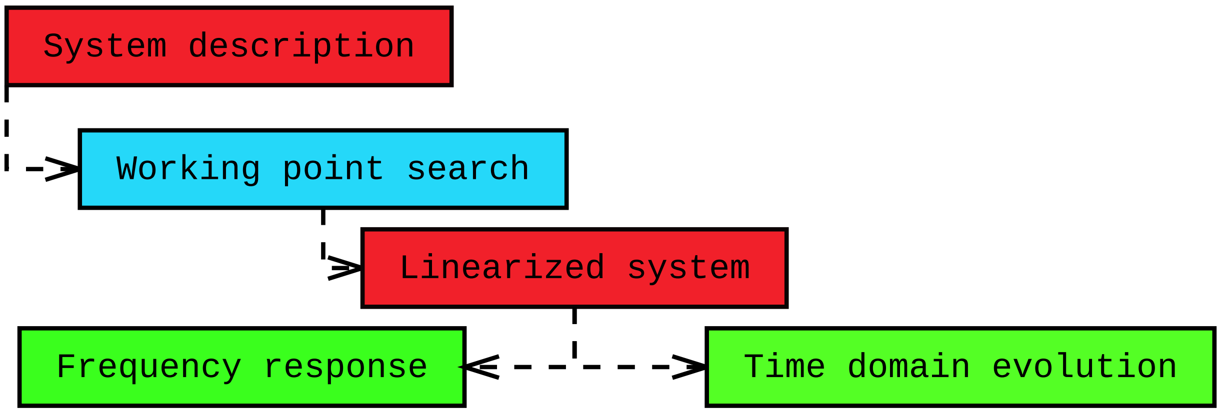
Each clamp fixes completely the relative displacement and orientation of frames.



- ✔ Only the relative constraints must be specified
- ✔ Optionally some (temptative) absolute frames positions can be given. This can help the search for the equilibrium position
- ✔ Force actuator apply forces between two objects (3 forces + 3 torques).
- ✔ Position actuator measures the relative position of two object (3 displacements + 3 rotations)

Simulation

A prerequisite is the search for the correct working point:



Coming back to the example.

```
while(t=pendulum.CurrentTime()<1.0) {
  coil1.set_fx(force(t));
  pendulum.TimeDynamics();
  cout << sensor.get_x();
}
```

or, in the frequency domain,

```
coil.set_fx(1.0,0.0);
coil.set_fy(0.0,1.0);
pendulum.FrequencyDynamics(f);
cout << sensor.get_x_mod();
```

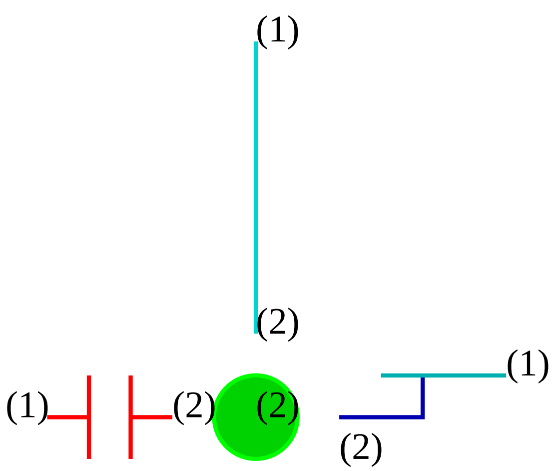
Simulation - system description

Each MObject must be able to provide:

1. A way to calculate the **static** forces on the frames, given their positions. This will be used in the working point search.
2. A **linearized** dynamic equation like

$$M\frac{d^2\vec{x}}{dt^2} + \Lambda\frac{d\vec{x}}{dt} + K\vec{x} = \vec{f}$$

   M,Λ and K are the mass,damping and stiffness arrays.
3. **Linear** relations between $\vec{x}$ and I/O variables (for an actuator or a sensor)



The system is partitioned in a collection of connected frames groups.

A reference frame is choosen in each group. This is optimized for numerical accuracy.

In this case:
Group 1: fixed inertial frame, no d.o.f.
Group 2: mass etc., 6 d.o.f.

Now we have a set of independent coordinates to describe system configuration.

Simulation - Nonlinearities

It is important to provide a non linearized static description. Look at the GAS blade:

$$W = \int_0^L \left[\frac{\gamma(l)}{2} \left(\frac{d\theta}{dl} \right)^2 - F_x \left(\cos\theta - \frac{x_0}{L} \right) - F_y \left(\sin\theta - \frac{y_0}{L} \right) \right] dl$$

The forces (and $\theta(l)$) are solution of the nonlinear problem

$$\frac{\delta W}{\delta \theta(l)} = 0, \; \frac{\delta W}{\delta F_x} = 0, \; \frac{\delta W}{\delta F_y} = 0$$

which can be, for example, discretized and solved numerically. The Blade object can do that, and the search for the working point can be schematized as follows:

1. Fix consistently the position of each frame
2. Ask each MObject to compute forces on frame
3. Compose these to find the ones coniugate to d.o.f.
4. Update d.o.f. (and frames) using some appropriate search algorithm
5. Go to point 2 until equilibrium is found

We end with F_x^{eq},F_y^{eq} and $\theta(l)^{eq}$

Simulation - time & frequency domain

When we know the working point, we get also a linearized description of each objects:

$$W^{lin} = \frac{1}{2} \int \int \frac{\delta^2 W}{\delta\theta(l)\delta\theta(l')}\bigg|_{\theta^{eq}} \delta\theta(l)\delta\theta(l')dldl'$$

- ✔ This is only a static description. We need dynamics. This problem can be solved adding a kinetic energy contribution. In the frequency domain this can be written as

$$T^{lin} = \int \int \frac{\omega^2}{2} M(l,l')\delta\theta(l)\delta\theta^\star(l')dldl'$$

- ✔ We want to write an expression which depends only on frames (boundary conditions). If we stay in the frequency domain this is simple. We can find the minimum of $T^{lin} - V^{lin}$, $\delta\theta^{lin}_{min}(l) = \delta\theta_0 f(l,\omega) + \delta\theta_L g(l,\omega)$,and substitute.

$$T^{lin} - W^{lin} = \frac{1}{2}\left(\delta\theta_0^\star \delta\theta_L^\star\right) \begin{pmatrix} L_{00}(\omega) & L_{01}(\omega) \\ L_{10(\omega)} & L_{11}(\omega) \end{pmatrix} \begin{pmatrix} \delta\theta_0 \\ \delta\theta_L \end{pmatrix}$$

We obtain our result, but L_{ij} are generally nonlinear functions of ω, and cannot be used easily in the time domain.

Low frequency approximation

We can expand last results in power of ω^2,

$$T^{lin} - W^{lin} = \frac{1}{2}\delta\vec{\theta}^{+} \cdot (-K + \omega^2 M + O(\omega^4)) \cdot \delta\vec{\theta}$$

defining our best candidate for the stiffness and mass array of the object.

- ✔ The same procedure can be applied to every mechanical objects.
- ✔ A natural classification of objects based on the number of specified (=not free) boundary conditions
- ✔ We expect accuracy at low frequencies

Now the mass,stiffness and damping array of the system can be constructed:

- ✘ Frequency domain: we solve $(-\omega^2 M - i\omega\Lambda + K)\vec{x} = \vec{f}$
- ✘ Time domain: the library provides several solvers:
  1. Standard Runge Kutta and Adams methods and st
  2. Methods based on state transition matrix

$$x(t+T) = e^{AT}\left(x(t) + \int_t^{t+T} e^{A(t-\tau)} f(\tau) d\tau\right)$$

  3. Methods for stiff systems

Longitudinal wire modes

We want an analytical solution, so we consider a straight wire. The longiudinal motion decouples:

$$T = \frac{1}{2}\int_0^L \rho S \left(\frac{dx}{dt}\right)^2 dl, \quad W = \frac{1}{2}\int_0^L ES \left(\frac{dx}{dl}\right)^2 dl$$

Neglecting T the potential energy is given simply by

$$W = \frac{1}{2}\frac{ES}{L}[x_L - x_0]^2$$

which is a simple spring. The $O(\omega^2)$can be obtained simply substituting the zero order solution in T. We get

$$T = \frac{1}{2}\frac{\rho SL}{3}[\dot{x}_0^2 + \dot{x}_0\dot{x}_L + \dot{x}_L^2]$$

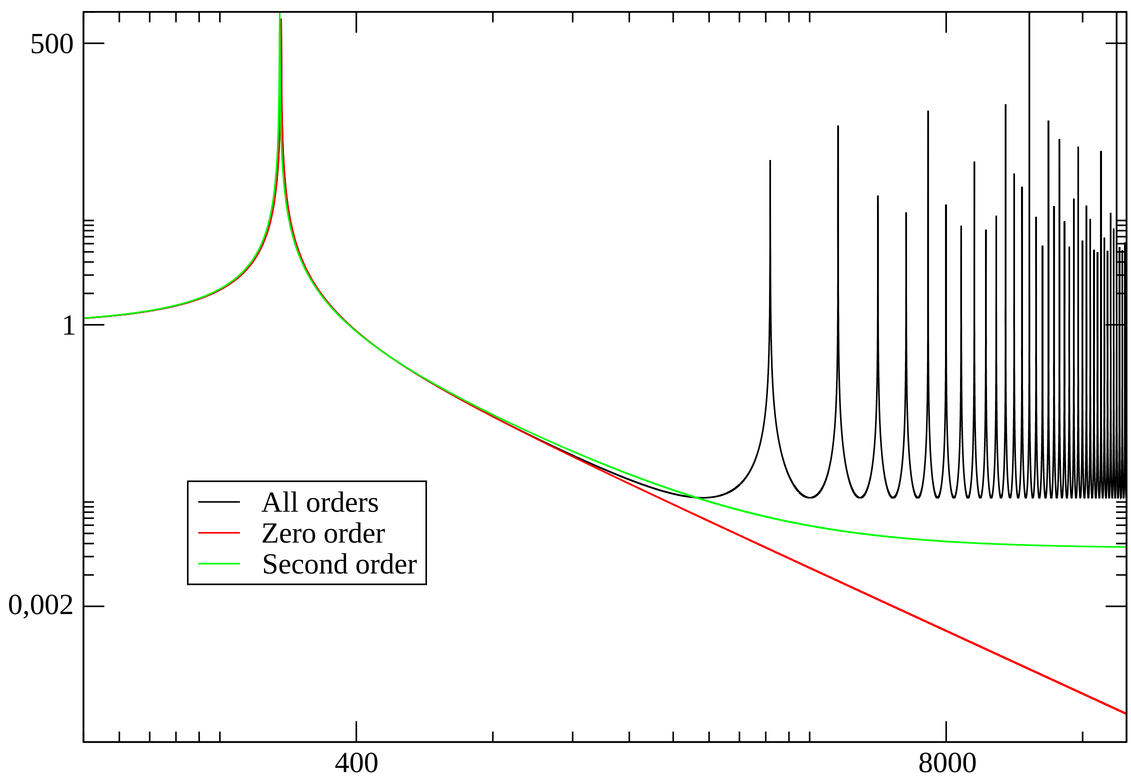
We can attach a mass and calculate the transfer function:

$$\frac{x_L}{x_0} = \frac{\frac{ES}{ML} + \omega^2\frac{\rho SL}{6M}}{\frac{ES}{ML} - \omega^2\left(1 + \frac{\rho SL}{3M}\right)}$$

A consistent mass matrix

In this case we can also calculate the exact transfer function

$$\frac{x_L}{x_0} = \frac{1}{\cos\frac{\omega^2 \rho L}{E} - \frac{M}{\rho S}\sin\frac{\omega^2 \rho L}{E}}$$



- ✔ The first kinetic correction can be important in many pratical cases
- ✔ It can be introduced without changing the d.o.f.

Cross coupling: the Beam case

We can apply the same procedure to a Beam.

✔ Two frames, so we expect a 12×12 mass, damping and stiffness array.

✔ In principle we must start from the appropriate potential, which is a quadratic function of the nonlinear stress tensor

$$u_{ij} = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i + \partial_i u_l \partial_j u_l\right)$$

In a case of pratical importance (small bending) it is possible to use a simpler approach.

1. We know 6 eigenvectors of the stiffness array (global traslation and rotation)
2. The reduced 6×6 array factorize in 2 2×2blocks (y, $-\theta_x$and x, θ_y) and 2 1×1blocks (θ_z,z).
3. For z we can use the longitudinal equation of the wire. The equation for θ_z is similar.
4. The 2×2 blocks can be calculated starting from

$$W_{trans} = \frac{1}{2}\int_0^L \left[EI\left(\frac{d^2y}{dl^2}\right)^2 + T\left(\frac{dy}{dl}\right)^2\right] dl$$

Spontaneous stiffness generation...

We obtain a 2×2 reduced stiffness array which can be written as

$$K(T) = \frac{T}{L(k^2 - 2\tanh(k/2))} \begin{pmatrix} 1 & -\frac{\tanh(k/2)}{k} \\ -\frac{\tanh(k/2)}{k} & \frac{1}{k}\left(\frac{1}{\tanh k} - \frac{1}{k}\right) \end{pmatrix}$$

where $k^2 = TL^2/EI$.

- ✔ This expression depends explicitly on T, because it is the result of a linearization around a stressed state
- ✔ We can solve this problem adding the longitudinal degree of freedom

  $$W = \frac{1}{2} q^T K\left(\frac{ES}{L}(z - L)\right) q + \frac{1}{2}\frac{ES}{L}(z - L)^2$$

  Note that this potential is nonlinear. But we know that

  $$ES(z_{eq} - L)/L \simeq T$$

  so the beam “get stiffnes” at the working point and we obtain an accurate linear approximation.
- ✔ The mass matrix can be evaluated without problems
- ✔ The general case is much more complicated. All the 6 independent degrees of freedom can be coupled together.

Internal modes

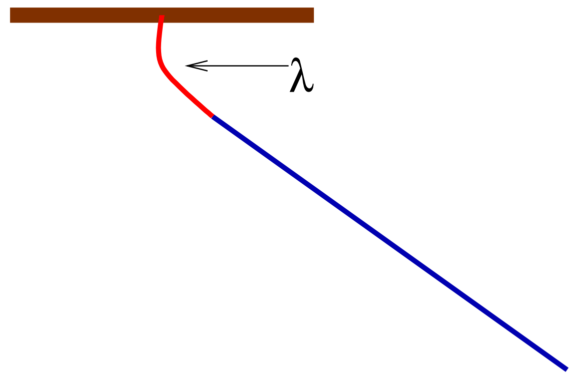
How to introduce internal modes, which are relevant in the high frequency region? There are many ways, the general idea is to add variables.

- ✔ The object (say, the wire) is seen as a collection of smallest ones. To each of them the low frequency approximation is applied.
- ✔ There are alternative approach, for example the finite element method
- ✔ In the MSE there is not a prescribed method to represent internal modes, it depends on the particular class implementation. The final result is always an “enlarged” stiffness (mass, damping) matrix of the following form

$$\begin{pmatrix} K_{00} & K_{0I} & 0 \\ K_{I0} & K_{II} & K_{I1} \\ 0 & K_{1I} & K_{11} \end{pmatrix}$$

  where the $0, 1$ indexes labels the frame variables, and I the internal,hidden ones.
- ✔ In this way we can avoid a complete three-dimensional representation of the internal modes when it is irrelevant (say, the torsional modes of a wire).

Singular perturbative problems



In the MSE the level of discretization of each mechanical object can be specified.

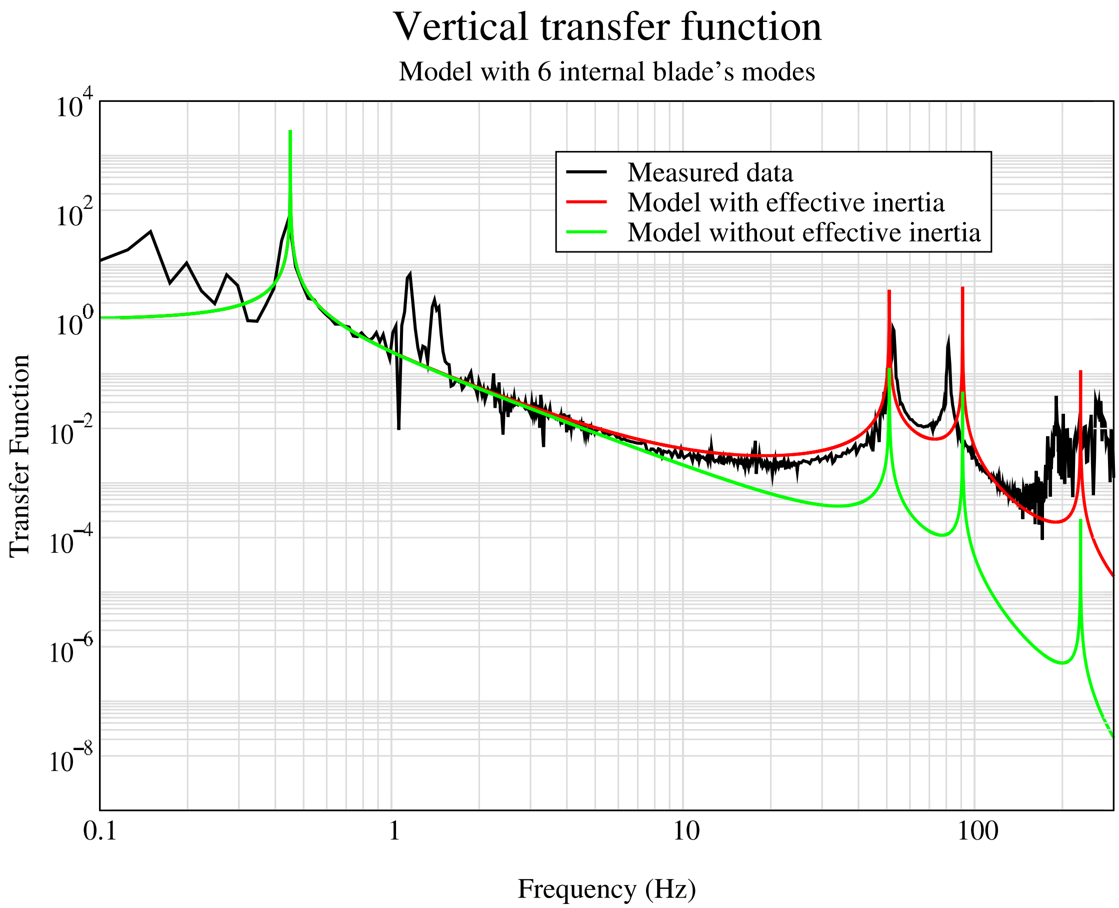
- ✔ Many elements give us many and more accurate internal modes. However the computational cost grows rapidly (roughly as N^3).
- ✔ In some cases it pays to use a more accurate procedure. Coming back to the beam equation, we can write

$$\frac{EI}{T} y'''' - y'' = 0$$

- ✔ Now suppose that $\lambda = \sqrt{EI/T}$ is small (as for a tensioned wire). The solution of $y'' = 0$ is not an accurate approximation around the suspension point.
- ✔ We have to adapt the discretization to this problem, in order to avoid an excessive number of elements, or inaccuracies.

An example: GAS filter

One of the first results:



- ✔ Single GAS filter with internal modes for the blade
- ✔ The addiction of the first kinetic correction give to the model a big improvement
- ✔ With 6 internal modes the first two internal resonances are in good agreement.

MSE integration in E2E

- ✔ The library can be used (and tested) independently, but its main purpose is the integration in the Ligo E2E model.
- ✔ The first stage is the construction of separate models, hard-wired to the E2E environment.
- ✔ In this way it will be possible to test the interaction between optics and mechanics
- ✔ In a future the library will be fully integrated. It will be possible to construct a mechanical system with the E2E GUI.

Actually the release 0.2 of MSE is avaliable.

1. It contains some basic mechanical objects that will be specialized in the near future
2. We are making extensive testing and debugging.
3. In a short time new classes for the description of composite system (a Ligo stack, a Geo triple pendulum, a GAS filter, an inverted pendum) will be implemented and tested

Future developments

1. **Thermal noise generation. This will be possible setting the temperature of the mechanical system. There are at least two possible approaches:**
   (a) **the addiction of stochastic forces of Langevin type**
   (b) **the use fluctuation-dissipation theorem to evaluate the relevant noise spectral densities.**
2. **Accurate modeling of structural damping. This will be done in two way:**
   (a) **using internal, hidden variables**
   (b) **projecting a viscous-type damping on the resonances**
3. **Adaptive meshing facilities**
4. **A “black box” class that will be used to introduce experimental data (say, a mechanical object with known transfer functions).**