

# SEI/SUS module for the LIGO end-to-end model: Formalism

Soumya D. Mohanty

This note describes the formalism I have developed for implementing the SEI/SUS module in the LIGO e2e model.

## I. OVERVIEW

The basic idea at the core of the formalism is the computation of action-reaction forces between the elements constituting the SEI/SUS. These forces are computed in terms of given external forces or displacements. The motion of each element (say, a wire, a spring or a rigid body) is solved, in a formal sense, in terms of the forces acting on the element. Some of these forces would be internal (i.e., action-reaction) and the rest would be external. One then computes, again formally, the acceleration of each point at which the element is connected to other elements in the system. The same exercise is carried out for each element and finally the acceleration of the points of attachment are equated. This then allows us to solve for the action-reaction forces in terms of given external forces and/or given displacements.

## II. SIMPLE EXAMPLES

I will illustrate the formalism by applying it to some simple systems.

### A. Simple pendulum

Consider a simple pendulum as shown in Fig. 1.

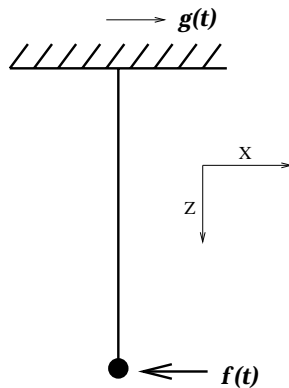


FIG. 1. The length of the string is  $l$  and the mass of the bob is  $m$ . The mass per unit length of the string is  $\rho$ .

I will assume that the transverse displacement of the string, denoted by  $u(z, t)$ , is confined to the  $XZ$  plane. I do not include the longitudinal mode of the string for the present. The driving force  $f(t)$  acts on the bob alone. A transverse force  $g(t)$  also acts at the suspension point of the string. I will compute the motion of the bob with the effect of the internal transverse modes of the string included. I assume viscous Damping specified by  $\gamma$ . The tension in the string (assumed constant along the string) is  $T$ .

The equation of motion for the string and the boundary conditions appropriate to this problem are,

$$\frac{\partial^2 u(z, t)}{\partial t^2} - \frac{T}{\rho} \frac{\partial^2 u(z, t)}{\partial z^2} + \frac{\gamma}{\rho} \frac{\partial u(z, t)}{\partial t} = 0, \quad (1)$$

$$-T \frac{\partial u(0, t)}{\partial z} = g(t); \quad T \frac{\partial u(l, t)}{\partial z} = r(t). \quad (2)$$

Here  $r(t)$  is the force exerted by the bob on the string and is the unknown force that needs to be computed in order to solve for the bob's motion. Without loss of generality, assume the initial position and velocity of the string to be zero at all points :

$$u(z, 0) = 0 ; \dot{u}(z, 0) = 0 . \quad (3)$$

The velocity of transverse waves  $a$  is of course given by,

$$a = \sqrt{\frac{T}{\rho}} . \quad (4)$$

We can formally obtain a solution for  $u(z, t)$  by using the Green's function  $G(z, \xi, t, \tau)$  and the standardising function  $w(z, t)$  for the above equation and boundary/initial conditions,

$$u(z, t) = \int_0^t d\tau \int_0^l d\xi G(z, \xi, t, \tau) w(\xi, \tau) , \quad (5)$$

$$w(z, t) = -a^2 \delta(z) \frac{g(t)}{T} + a^2 \delta(z - l) \frac{r(t)}{T} . \quad (6)$$

Thus, we obtain,

$$u(z, t) = -\frac{a^2}{T} \int_0^t d\tau G(z, 0, t, \tau) g(\tau) + \frac{a^2}{T} \int_0^t d\tau G(z, l, t, \tau) r(\tau) . \quad (7)$$

Now, the wave equation is time translation invariant. Therefore,  $G(z, \xi, t, \tau) = G(z, \xi, t - \tau)$ . Thus it makes sense to talk about the Laplace transform of the Green's function here.

Let the Laplace transforms of  $G(z, \xi, t - \tau)$  with respect to the time argument be  $W(z, \xi, s)$ . Let the Laplace transform of  $u(z, t)$ ,  $g(t)$ ,  $r(t)$  and  $f(t)$  be  $U(z, s)$ ,  $G(s)$ ,  $R(s)$  and  $F(s)$  respectively. From Eq. (7), we obtain

$$U(z, s) = -\frac{a^2}{T} W(z, 0, s) G(s) + \frac{a^2}{T} W(z, l, s) R(s) . \quad (8)$$

The Laplace transform of  $\ddot{u}(z, t)$  is given by  $s^2 U(z, s)$ .

Now comes the crucial step in the formalism. Since the bob and string are attached at all times, the bob and the end of the string attached to it *must* have the same acceleration. Let the transverse displacement of the bob be  $x_b(t)$ . Then,

$$m\ddot{x}_b = f(t) - r(t) . \quad (9)$$

Equating the acceleration of the bob and the string at  $z = l$ , we obtain in the Laplace domain,

$$s^2 \left[ -\frac{a^2}{T} W(l, 0, s) G(s) + \frac{a^2}{T} W(l, l, s) R(s) \right] = \frac{F(s) - R(s)}{m} . \quad (10)$$

Eq. (10) can be solved for  $R(s)$  yielding,

$$R(s) = \frac{F(s)/m + s^2 a^2 W(l, 0, s) G(s)/T}{1/m + s^2 a^2 W(l, l, s)/T} . \quad (11)$$

It is possible to get closed form expressions for  $W(l, 0, s)$  and  $W(l, l, s)$ . From a standard table [1], we get

$$W(l, 0, s) = \frac{1}{a^2 s' \sinh(s'l)} , \quad (12)$$

$$W(l, l, s) = \frac{1}{a^2 s' t \cosh(s'l)} , \quad (13)$$

$$s' = \frac{1}{a} \sqrt{s^2 + 2s\gamma/\rho} . \quad (14)$$

$$(15)$$

One would expect that the transfer functions obtained above would include the pendulum mode. However, this is not so. Initially this was a bit puzzling until I realized that such a solution is obtained for a different set of boundary conditions where the displacement of the string at  $z = 0$  is forced to be zero (or a specified function of time).

Let the motion of the string at the suspension point be a given function  $x_s(t)$ . Then the force  $g(t)$  must be such that the displacement  $u(0, t) = x_s(t)$ . This implies that (using Eq. (8)),

$$G(s) = - \frac{TX_s(s)/a^2 - W(0, l, s)R(s)}{W(0, 0, s)}, \quad (16)$$

where  $X_s(s)$  is the Laplace transform of  $z_s(t)$ . From the symmetry of Green's function,

$$W(0, l, s) = W(l, 0, s), \quad (17)$$

and it can be shown that, in the present case,

$$W(l, l, s) = W(0, 0, s). \quad (18)$$

Substituting for  $G(s)$  in Eq. (10) and using the above relations, we get,

$$R(s) = \frac{F(s) - ms^2 X_s(s)W(l, 0, s)/W(0, 0, s)}{1 + ms^2 a^2 [W(l, l, s) - W(l, 0, s)W(0, l, s)/W(0, 0, s)]/T}, \quad (19)$$

$$= \frac{F(s) - ms^2 X_s(s)/\cosh(s'l)}{1 + ms^2 \tanh(ls')/(Ts')}. \quad (20)$$

I will now work out the motion of the bob in some limiting situations as a check on the above analysis. (a) Let  $x_s(t) = 0$  and  $\gamma = 0$ . Then  $s' = s/a$  and

$$ms^2 X_b(s) = F(s) \left[ 1 - \frac{1}{1 + msa \tanh(ls/a)/T} \right]. \quad (21)$$

In the Fourier domain, the transfer function becomes<sup>1</sup>,

$$\frac{\tilde{X}_b(\omega)}{\tilde{F}(\omega)} = \frac{1}{\omega T} \left[ \frac{a \tan(l\omega/a)}{1 - m\omega a \tan(l\omega/a)/T} \right]. \quad (22)$$

## B. Simple pendulums in series

I will now consider a more complicated system which will illustrate the method that will be followed for the SEI/SUS model. Consider the system shown in Fig. 2.

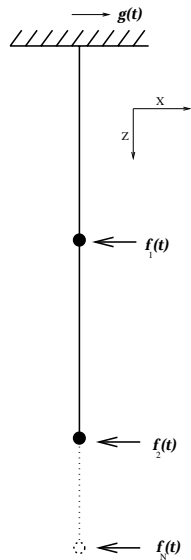


FIG. 2. Simple pendulums in series. The mass of each bob is kept the same for simplicity.

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<sup>1</sup>The transfer function, being causal in the time domain, is zero for  $t < 0$ . Hence, the transfer function in the Fourier domain is obtained by  $s \rightarrow i\omega$ .

The equation of motion of the  $j^{\text{th}}$  string and its boundary conditions are given by,

$$\frac{\partial^2 u_j(z, t)}{\partial t^2} - \frac{T_j}{\rho} \frac{\partial^2 u_j(z, t)}{\partial z^2} + \frac{\gamma}{\rho} \frac{\partial u_j(z, t)}{\partial t} = 0, \quad (23)$$

$$-T_j \frac{\partial u_j((j-1)l, t)}{\partial z} = r_{j,j-1}(t); \quad T_j \frac{\partial u_j(jl, t)}{\partial z} = r_{j,j}(t), \quad (24)$$

where  $r_{i,j}(t)$  is the reaction of the  $j^{\text{th}}$  bob on the  $i^{\text{th}}$  string and  $r_{10}(t)$  is understood to be the same as  $g(t)$ . The equation of motion of the  $k^{\text{th}}$  bob is,

$$m\ddot{x}_k = f_k(t) - r_{k,k}(t) - r_{k+1,k}(t). \quad (25)$$

The initial positions and velocities are assumed to be zero.

As in the previous example,  $u_j(z, t)$  can be formally solved in terms of the reaction forces. For the  $j^{\text{th}}$  string we get,

$$U_j(z, s) = -\frac{a^2}{T_j} W_{j,0}(z, s) R_{j,j-1}(s) + \frac{a^2}{T_j} W_{j,l}(z, s) R_{j,j}(s). \quad (26)$$

$$W_{j,0}(z, s) = W(z - (j-1)l, 0, s), \quad (27)$$

$$W_{j,l}(z, s) = W(z - (j-1)l, l, s), \quad (28)$$

Where  $W(z, \xi, s)$  is as defined earlier.

I then equate the acceleration of the  $k^{\text{th}}$  bob with those of the end points of each of the two strings attached to it. Thus, I get two acceleration balance equations for the  $j^{\text{th}}$  bob when  $j \neq N$ ,

$$s^2 \left[ -\frac{a^2}{T_j} W_{j,0}(jl, s) R_{j,j-1}(s) + \frac{a^2}{T_j} W_{j,l}(jl, s) R_{j,j}(s) \right] = \frac{F_j(s) - R_{j,j}(s) - R_{j+1,j}(s)}{m}, \quad (29)$$

$$s^2 \left[ -\frac{a^2}{T_{j+1}} W_{j+1,0}(jl, s) R_{j+1,j}(s) + \frac{a^2}{T_{j+1}} W_{j+1,l}(jl, s) R_{j+1,j+1}(s) \right] = \frac{F_j(s) - R_{j,j}(s) - R_{j+1,j}(s)}{m}. \quad (30)$$

For  $j = N$ , we get a single equation,

$$s^2 \left[ -\frac{a^2}{T_N} W_{N,0}(Nl, s) R_{N,N-1}(s) + \frac{a^2}{T_N} W_{N,l}(Nl, s) R_{N,N}(s) \right] = \frac{F_N(s) - R_{N,N}(s)}{m}. \quad (31)$$

Eq. (29) and Eq. (30) can be recast as follows. First, write Eq. (29) and Eq. (30) as a single matrix equation,

$$\begin{pmatrix} \frac{s^2 a^2}{T_j} W_{j,l}(jl, s) + \frac{1}{m} & \frac{1}{m} \\ \frac{1}{m} & -\frac{s^2 a^2}{T_{j+1}} W_{j+1,l}(jl, s) + \frac{1}{m} \end{pmatrix} \begin{pmatrix} R_{j,j} \\ R_{j+1,j} \end{pmatrix} + \begin{pmatrix} -\frac{s^2 a^2}{T_j} W_{j,0}(jl, s) R_{j,j-1} \\ \frac{s^2 a^2}{T_{j+1}} W_{j+1,0}(jl, s) R_{j+1,j+1} \end{pmatrix} = \frac{F_j}{m} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (32)$$

By taking the scalar product of the above expression with, first,  $(1 \ 1)$  and then  $(1 \ -1)$ , we get,

$$\begin{pmatrix} -\frac{s^2 a^2}{T_j} W_{j,0}(jl, s) & \frac{s^2 a^2}{T_j} W_{j,l}(jl, s) + \frac{2}{m} & -\frac{s^2 a^2}{T_{j+1}} W_{j+1,l}(jl, s) + \frac{2}{m} & \frac{s^2 a^2}{T_{j+1}} W_{j+1,0}(jl, s) \end{pmatrix} \begin{pmatrix} R_{j,j-1} \\ R_{j,j} \\ R_{j+1,j} \\ R_{j+1,j+1} \end{pmatrix} = \frac{F_j}{m}, \quad (33)$$

$$\begin{pmatrix} -\frac{s^2 a^2}{T_j} W_{j,0}(jl, s) & \frac{s^2 a^2}{T_j} W_{j,l}(jl, s) & \frac{s^2 a^2}{T_{j+1}} W_{j+1,l}(jl, s) & -\frac{s^2 a^2}{T_{j+1}} W_{j+1,0}(jl, s) \end{pmatrix} \begin{pmatrix} R_{j,j-1} \\ R_{j,j} \\ R_{j+1,j} \\ R_{j+1,j+1} \end{pmatrix} = 0. \quad (34)$$

For  $j = 1$ , Eq. (33) and Eq. (34) are slightly different,

$$\begin{pmatrix} \frac{s^2 a^2}{T_1} W_{1,l}(l, s) + \frac{2}{m} & -\frac{s^2 a^2}{T_2} W_{2,l}(l, s) + \frac{2}{m} & \frac{s^2 a^2}{T_2} W_{2,0}(l, s) \end{pmatrix} \begin{pmatrix} R_{1,1} \\ R_{2,1} \\ R_{2,2} \end{pmatrix} = \frac{F_1}{m} + \frac{s^2 a^2}{T_1} W_{1,0}(l, s) G(s), \quad (35)$$

$$\begin{pmatrix} \frac{s^2 a^2}{T_1} W_{1,l}(l, s) & \frac{s^2 a^2}{T_2} W_{2,l}(l, s) & -\frac{s^2 a^2}{T_2} W_{2,0}(l, s) \end{pmatrix} \begin{pmatrix} R_{1,1} \\ R_{2,1} \\ R_{2,2} \end{pmatrix} = \frac{s^2 a^2}{T_1} W_{1,0}(l, s) G(s). \quad (36)$$

Collecting all the acceleration balance equations together gives a matrix equation of the form,

$$\mathbf{T}\mathbf{R} = \frac{1}{m}\mathbf{F}, \quad (37)$$

$$\mathbf{R} = [R_{1,1}(s), \dots, R_{j,j-1}(s), R_{j,j}(s), R_{j+1,j}(s), R_{j+1,j+1}(s), \dots, R_{N,N}(s)]^T, \quad (38)$$

$$\mathbf{F} = \left[ F_1(s) + \frac{s^2 a^2 m}{T_1} W_{1,0}(l, s) G(s), \frac{s^2 a^2 m}{T_1} W_{1,0}(l, s) G(s), F_2(s), 0, F_3(s), 0, \dots, 0, F_N(s) \right]^T, \quad (39)$$

and  $\mathbf{T}$  is a matrix containing all the  $W$ 's. Given  $\mathbf{F}$ , one can solve for  $\mathbf{R}$ ,

$$\mathbf{R} = \frac{1}{m}\mathbf{T}^{-1}\mathbf{F}. \quad (40)$$

Using the reaction forces, one can then solve for the motion of each bob in the Laplace domain.

For  $k = 1, 2, \dots, N - 2$ , the non-zero elements of row  $2k + 1$  of  $\mathbf{T}$  are given by the row vector on the LHS of Eq. (33) with  $j$  substituted by  $k + 1$ . Similarly, the non-zero elements of row  $2(k + 1)$  are given by Eq. (34), again with  $j = k + 1$ . For every  $k$ , the first non-zero element of row  $2k + 1$  and row  $2(k + 1)$  occurs in column  $2k$  of  $\mathbf{T}$  and the last non-zero element occurs in column  $2k + 3$ . The first and second rows of  $\mathbf{T}$  are given by the row vectors in Eq. (35) and Eq. (36) respectively. The first non-zero element in both rows occurs in column 1 and the last non-zero element occurs in column 3. From Eq. (31), we obtain the last row of  $\mathbf{T}$ ,

$$\left( 0 \ 0 \ \dots \ -\frac{s^2 a^2}{T_N} W_{N,0}(Nl, s) \ \frac{s^2 a^2}{T_N} W_{N,l}(Nl, s) + \frac{1}{m} \right). \quad (41)$$

Pictorially, the matrix  $\mathbf{T}$  would look somewhat like the schematic in Fig. 3. As one would expect, most of the matrix is zero.

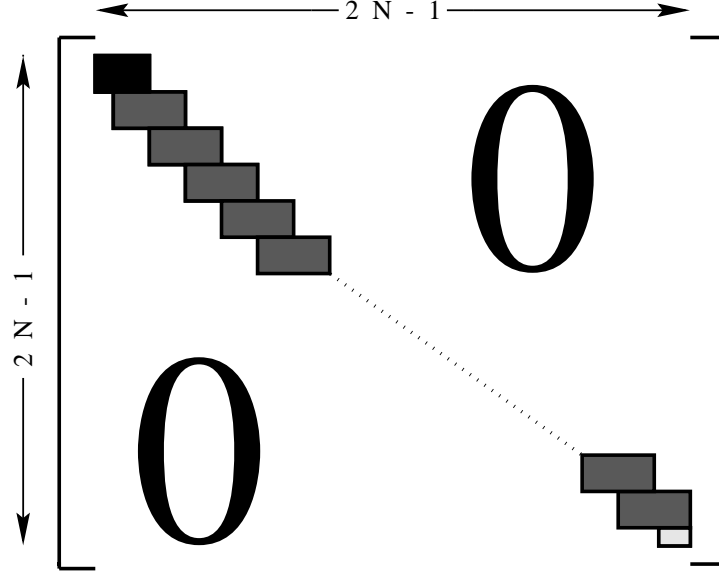


FIG. 3. The structure of  $\mathbf{T}$ . Each gray box is a  $2 \times 4$  matrix. The black box is a  $2 \times 3$  while the unfilled box is a  $1 \times 2$  matrix.

### III. THE FORMALISM

In the modelling of the SEI/SUS system, we will use the same technique as above. That is, (1) obtain a formal solution for the motion of each element in terms of the forces, action-reaction as well as external, acting on it. (2) From these solutions, find the accelerations of the points at which the elements are attached to one another. (3) Equate the accelerations for each point. This provides an equation such as Eq. (37) which should be solved for  $\mathcal{R}$ .

From our experience with the previous example, we can say that the matrix  $\mathbf{T}$  would, in general, be a sparse matrix. This is because the acceleration balance equations for any one point of attachment would involve only a *proper* subset of the full set  $\mathcal{R}$  of action-reaction forces. Hence, the row of  $\mathbf{T}$

that corresponds to this point of attachment would have all elements zero apart from the ones that pick up the proper subset of  $\mathcal{R}$  that is involved. However, it is difficult to say whether this sparse matrix would have a nice pattern as in the previous example. If it does have a special pattern, there exist special techniques for inversion of such matrices that are quite efficient.

For any form of  $\mathbf{T}$ , my approach would be to invert  $\mathbf{T}$  using a symbolic algebra program. Each row of  $\mathbf{T}^{-1}$  is a collection of transfer functions in the Laplace domain. It should be possible to convert them into digital filters. Thus, all the action-reaction forces would finally be obtained as digital filters acting on the given external forces or displacements in the time domain.

It should be noted this formalism can handle only linear systems, i.e., systems for which the equations of motions for each element are linear differential equations. All that is required for such systems is the set of Green's functions that describe each element. These can be computed before hand either numerically or algebraically and stored as separate modules. Once a system design has been specified, these modules can be connected together to yield  $\mathbf{T}$ . I believe that the entire process, for a given design specification, can be automated.

I now describe an algorithm that can be used for setting up the matrix  $\mathbf{T}$  in Eq. (37). It should be noted that many alternatives to this algorithm are possible. The steps are as follows.

(1) Fix a global cartesian reference frame (GRF). This frame is global in the sense that it describes an entire SEI/SUS system at any one location in the interferometer. There may exist a frame that describes the entire interferometer itself. Such a frame would be related to the GRF frame via a fixed rotation matrix.

(2) Consider the state of mechanical equilibrium for the entire system. In this state record the locations of all the points of attachments between elements and also, for each element, the rotational transformation needed to go from the GRF frame to the *preferred frame* (see below) of that element.

(3) Each element has a preferred frame (cartesian) in which its Green's functions are expressible in the simplest manner. For instance, for a wire the preferred frame is one where the wire lies along one of the axes. (Say, the  $Z$  axis as in the previous examples.) Let the rotational transformation from the GRF frame to the preferred frame for the  $i^{\text{th}}$  element be denoted by  $\mathcal{R}_i$ . For every element, transform the GRF frame components of the external forces acting on it into the preferred frame. The preferred frame Green's functions would be computed and stored beforehand. Formally solve for the acceleration of all the points of attachments of a given element in the Laplace domain. Transform the acceleration components back into the GRF frame.

(4) So now we have the GRF frame components of the acceleration of each point of attachment. Apply the equality of acceleration condition for each point of attachment and transfer terms containing the action-reaction force terms to the LHS and terms containing driving forces to the RHS.

(5) The set of algebraic equations so obtained then furnishes the matrix  $\mathbf{T}$ .

We will now compute the preferred frame formal solutions for accelerations of points of attachments for several elements out of which a model of the LIGO SEI/SUS system may be built.

## A. A straight wire

### 1. One transverse polarization, no longitudinal mode, viscous damping

This case has already been discussed (see Section II A). Here the relevant expression is the one given in Eq. (8) which can be used to obtain the accelerations at the two points of attachments.

### 2. One transverse polarization, no longitudinal mode, internal damping

The equation of motion of the wire [2] is,

$$\frac{\partial^2 u(z, t)}{\partial t^2} - a^2 \frac{\partial^2 u(z, t)}{\partial z^2} + \frac{EI}{T} \frac{\partial^4 u(z, t)}{\partial z^4} = 0, \quad (42)$$

where  $E$  is the Young's modulus and  $I$  is the area moment of inertia. with boundary conditions,

$$EI \frac{\partial^3 u(0, t)}{\partial z^3} - T \frac{\partial u(0, t)}{\partial z} = g(t), \quad (43)$$

$$EI \frac{\partial^3 u(l, t)}{\partial z^3} - T \frac{\partial u(l, t)}{\partial z} = -r(t). \quad (44)$$

We assume the initial conditions  $u(z, 0) = \dot{u}(z, 0) = 0$ . To introduce damping in the wire,  $E$  will be made complex in the final transfer function.

3. *Two transverse polarizations, no longitudinal mode and viscous damping*

$$\frac{\partial^2 u_x}{\partial t^2} - a_x^2 \frac{\partial^2 u_x}{\partial z^2} + \gamma_x \frac{\partial u_x}{\partial t} = 0, \quad (45)$$

$$-T \frac{\partial u_x(0, t)}{\partial z} = g_x(t); \quad T \frac{\partial u_x(l, t)}{\partial z} = r_x(t), \quad (46)$$

$$\frac{\partial^2 u_y}{\partial t^2} - a_y^2 \frac{\partial^2 u_y}{\partial z^2} + \gamma_y \frac{\partial u_y}{\partial t} = 0 \quad (47)$$

$$-T \frac{\partial u_y(0, t)}{\partial z} = g_y(t); \quad T \frac{\partial u_y(l, t)}{\partial z} = r_y(t), \quad (48)$$

The required values of the Green's function for each degree of freedom have already been provided in Eq. (12) and Eq. (13). However,  $\gamma/\rho$  should be replaced by  $\gamma_x$  or  $\gamma_y$  as appropriate and so should  $a$  in those expressions. We attach the appropriate subscript, 'x' or 'y', to distinguish between the two Green's functions and their Laplace transforms. Thus, from Eq. (8), we get,

$$U_x(z, s) = -\frac{a_x^2}{T} W_x(z, 0, s) G_x(s) + \frac{a_x^2}{T} W_x(z, l, s) R_x(s), \quad (49)$$

$$U_y(z, s) = -\frac{a_y^2}{T} W_y(z, 0, s) G_y(s) + \frac{a_y^2}{T} W_y(z, l, s) R_y(s). \quad (50)$$

Using the above expressions we can write down the acceleration at the end points.

4. *Two transverse polarizations, longitudinal mode and viscous damping*

As discussed above, in the small deformation approximation, the three degrees of freedom can be decoupled. Hence, the equations of motion for the transverse polarizations are the same as before (see Eq. (49) and Eq. (50)). For the longitudinal mode, we have

$$\frac{\partial^2 u_z}{\partial t^2} - a_z^2 \frac{\partial^2 u_z}{\partial z^2} + \gamma_z \frac{\partial u_z}{\partial t} = 0, \quad (51)$$

$$-Y \frac{\partial u_z(0, t)}{\partial z} = g_z(t); \quad Y \frac{\partial u_z(l, t)}{\partial z} = r_z(t) \quad (52)$$

where  $Y = \mathcal{Y}\pi r_{\text{wire}}^2$ ,  $r_{\text{wire}}$  being the radius of the wire and  $\mathcal{Y}$  is its Young's modulus. In the Laplace domain,

$$U_z(z, s) = -\frac{a_z^2}{Y} W_z(z, 0, s) G_z(s) + \frac{a_z^2}{Y} W_z(z, l, s) R_z(s). \quad (53)$$

5. *Two transverse polarizations, spring mode and internal damping*

**B. A rigid body**

We consider the case of small angular motions for which Euler's equations for the motion of a rigid body can be linearised. Euler's equations evolve the angles that describe the rotation from a fixed orientation frame attached to the body's centre of mass, say  $S$ , to the body axes that rotate with the body (say, frame  $B$ ). The angles used here are defined in Appendix A. We begin with a description of the various reference frames that will be needed here apart from the GRF.

Let the body frame  $B$  be attached to the centre of mass (CM) of the body and oriented along its principal axes. Let  $I_1$ ,  $I_2$  and  $I_3$  be the principal moments of inertia. Determine the orientation of  $B$  when the system is in mechanical equilibrium. We have already mentioned that  $S$  is attached to the CM. However, it has a fixed orientation. We orient our space axes to match the body axes *at equilibrium*. Let the rotational transform *from* GRF *to*  $S$  be denoted by  $\mathcal{R}$  (which is a constant matrix). Let the forces on the body be  $(\bar{r}_1, \bar{r}_2, \dots, \bar{r}_n)$ , the set of action-reaction forces, and  $(\bar{f}_1, \bar{f}_2, \dots, \bar{f}_m)$ , the set of external forces (such as control forces). The equation of motion for the centre of mass in the GRF is

$$M \ddot{\mathbf{x}}_{\text{cm}} = \sum_{i=1}^m \bar{f}_i + \sum_{i=1}^n \bar{r}_i, \quad (54)$$

where  $M$  is the body's mass and  $\bar{\mathbf{x}}_{\text{cm}}$  is the position of the centre of mass.

The rotational transformation from  $B$  to  $S$  at any instant is denoted here by  $R_{b2s}$ ,

$$R_{b2s} = \mathbf{1} + \epsilon, \quad (55)$$

$$\epsilon = \begin{pmatrix} 0 & -\phi & \eta \\ \phi & 0 & -\theta \\ -\eta & \theta & 0 \end{pmatrix}. \quad (56)$$

For the derivation and definition of the angles used here, see Appendix A. For any vector  $\bar{\mathbf{X}}$ ,

$$[\bar{\mathbf{X}}]_B = R_{b2s}^T [\bar{\mathbf{X}}]_S. \quad (57)$$

The components of a vector in the  $S$  frame are obtained from the components in the GRF frame as,

$$[\bar{\mathbf{X}}]_S = \mathcal{R} [\bar{\mathbf{X}}]_{\text{GRF}}. \quad (58)$$

Note that since  $\epsilon$  is an antisymmetric matrix,

$$\epsilon^T \bar{\mathbf{X}} \equiv \bar{\mathbf{e}} \times \bar{\mathbf{X}}, \quad (59)$$

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} := - \begin{pmatrix} \theta \\ \eta \\ \phi \end{pmatrix}. \quad (60)$$

Let the  $r$ -forces act at points  $\bar{\mathbf{p}}_1, \dots, \bar{\mathbf{p}}_n$  respectively and let the  $f$ -forces act at points  $\bar{\mathbf{P}}_1, \dots, \bar{\mathbf{P}}_m$ , where these position vectors are measured from the CM. If we assume that the points at which forces are applied are fixed with respect to  $B$ , the above position vectors would have constant components in  $B$ . Let the torques, about the CM, due to the  $r$ -forces be  $\bar{\boldsymbol{\tau}}_i$  and those due to the  $f$ -forces be  $\bar{\boldsymbol{\tau}}_i$ . Let the sum of all these torques be  $\bar{\mathbf{N}}$ . Using Eq. (55), Eq. (57) and Eq. (59) we get

$$\begin{aligned} [\bar{\mathbf{N}}]_B &= \sum_{i=1}^n [\bar{\mathbf{p}}_i]_B \times (R_{b2s}^T [\bar{\boldsymbol{\tau}}_i]_S) + \sum_{i=1}^m [\bar{\mathbf{P}}_i]_B \times (R_{b2s}^T [\bar{\boldsymbol{\tau}}_i]_S), \\ &= \sum_{i=1}^n [\bar{\mathbf{p}}_i]_B \times [\bar{\boldsymbol{\tau}}_i]_S + \sum_{i=1}^m [\bar{\mathbf{P}}_i]_B \times [\bar{\boldsymbol{\tau}}_i]_S + \mathbf{A} \bar{\mathbf{e}}, \end{aligned} \quad (61)$$

$$\mathbf{A}_{kl} = \sum_{i=1}^n [([\bar{\mathbf{p}}_i]_B \cdot [\bar{\boldsymbol{\tau}}_i]_S) \delta_{ij} - [\bar{\mathbf{p}}_i]_{B,k} [\bar{\boldsymbol{\tau}}_i]_{S,l}] + \sum_{i=1}^m [([\bar{\mathbf{P}}_i]_B \cdot [\bar{\boldsymbol{\tau}}_i]_S) \delta_{ij} - [\bar{\mathbf{P}}_i]_{B,k} [\bar{\boldsymbol{\tau}}_i]_{S,l}], \quad (62)$$

where  $[\bar{\mathbf{X}}]_{Y,j}$  denotes the  $j^{\text{th}}$  component of  $\bar{\mathbf{X}}$  in the frame  $Y$ . We can neglect the last term in Eq. (61) since it is at a lower order in the angles with respect to the other terms. The latter just approximate the  $B$  frame torque by keeping the  $B$  frame forces the same as the  $S$  frame ones. It is possible that for some forces the first order term dominates over the zeroth order ones. However, that would imply that such forces produce very small torques compared to the rest and hence will not affect the rotational dynamics much. Thus, we get for Euler's equations (see Eq. (A16) – Eq. (A18))

$$\begin{pmatrix} I_1 \ddot{\theta} \\ I_2 \ddot{\eta} \\ I_3 \ddot{\phi} \end{pmatrix} = \sum_{i=1}^n [\bar{\mathbf{p}}_i]_B \times [\bar{\boldsymbol{\tau}}_i]_S + \sum_{i=1}^m [\bar{\mathbf{P}}_i]_B \times [\bar{\boldsymbol{\tau}}_i]_S. \quad (63)$$

The above equations can be solved in the Laplace domain. Let the Laplace transforms of the angles be denoted by placing a  $\hat{\cdot}$  on top of the corresponding time domain symbol. Then<sup>2</sup>,

$$\begin{pmatrix} \hat{\theta} \\ \hat{\eta} \\ \hat{\phi} \end{pmatrix} = \frac{1}{s^2} \begin{pmatrix} I_1^{-1} & 0 & 0 \\ 0 & I_2^{-1} & 0 \\ 0 & 0 & I_3^{-1} \end{pmatrix} \left[ \sum_{i=1}^n [\bar{\mathbf{p}}_i]_B \times [\bar{\boldsymbol{\tau}}_i(s)]_S + \sum_{i=1}^m [\bar{\mathbf{P}}_i]_B \times [\bar{\boldsymbol{\tau}}_i(s)]_S \right]. \quad (64)$$

Now, we need to obtain the acceleration of each of the points  $\bar{\mathbf{p}}_j$ . The GRF components of  $\bar{\mathbf{p}}_j$  are given by,

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<sup>2</sup>Assuming as before that the initial values of the angles and angular velocities are zero.

$$\begin{aligned}
[\bar{p}_j]_{\text{GRF}} &= \mathcal{R}^T R_{b2s} [\bar{p}_j]_B + \bar{x}_{\text{cm}} , \\
&= \mathcal{R}^T [\bar{p}_j]_B - \mathcal{R}^T [\bar{e} \times [\bar{p}_j]_B] + \bar{x}_{\text{cm}} .
\end{aligned} \tag{65}$$

In terms of Laplace transforms,

$$[\bar{p}_j(s)]_{\text{GRF}} = \frac{1}{s} \mathcal{R}^T [\bar{p}_j]_B + \mathcal{R}^T \left[ \begin{pmatrix} \bar{\theta} \\ \bar{\eta} \\ \bar{\phi} \end{pmatrix} \times [\bar{p}_j]_B \right] + \bar{X}_{\text{CM}}(s) , \tag{66}$$

In the Laplace domain, the acceleration is given by ( using Eq. (54)),

$$\begin{aligned}
s^2 [\bar{p}_j(s)]_{\text{GRF}} - s [\mathcal{R}^T [\bar{p}_j]_B + \bar{x}_{\text{cm}}(0)] &= \sum_{i=1}^n \mathcal{R}^T \mathbf{H}_1(i, j) [\bar{R}_i(s)]_S + \sum_{i=1}^m \mathcal{R}^T \mathbf{H}_2(i, j) [\bar{F}_i(s)]_S + \\
&\quad \frac{1}{M} \left[ \sum_{i=1}^n \bar{R}_i(s) + \sum_{i=1}^m \bar{F}_i(s) \right] ,
\end{aligned} \tag{67}$$

$$\mathbf{H}_1(i, j) = \begin{pmatrix} p_{i,2}p_{j,2}/I_3 + p_{i,3}p_{j,3}/I_2 & -p_{i,1}p_{j,2}/I_3 & -p_{i,1}p_{j,3}/I_2 \\ -p_{j,1}p_{i,2}/I_3 & p_{i,1}p_{j,1}/I_3 + p_{i,3}p_{j,3}/I_1 & -p_{i,2}p_{j,3}/I_1 \\ -p_{j,1}p_{i,3}/I_2 & -p_{j,2}p_{i,3}/I_1 & p_{i,1}p_{j,1}/I_2 + p_{i,2}p_{j,2}/I_1 \end{pmatrix} , \tag{68}$$

where  $p_{i,m}$  stands for the  $m^{\text{th}}$  component of  $[\bar{p}_i]_B$  and similarly for  $p_{j,n}$ . The matrix  $\mathbf{H}_2(i, j)$  can be obtained by the replacement  $p_{i,m} \rightarrow P_{i,m}$  and leaving  $p_{j,m}$  the same. **Eq. (67) is the key equation for this module.**

### 1. An example : viscously damped suspension wires and cylindrical mirror

Consider the model shown in Fig. 4 for the LIGO end mirror suspension. Note that instead of a single loop of wire we have modelled the suspension as having two independent wires. We assume that all those symmetries are present in the system that are required to make the body axes as well as the space frame axes lie, at equilibrium, along the  $\mathbf{XYZ}$  axes shown. However, the origins of the  $B$  and  $S$  frame are attached to the CM. The GRF is also assumed to be the one shown in the figure ( $\Rightarrow \mathcal{R} = \mathbf{1}$ ).

The list of  $\mathbf{H}_1(i, j)$  and  $\mathbf{H}_2(i, j)$  (see Eq. (67) and Eq. (68)) matrices are given in Appendix B. Since the  $S$  frame here is taken to be the same as the GRF, the acceleration of the attachment point of the  $j^{\text{th}}$  wire is given by

$$\sum_{i=1}^2 \mathbf{H}_1(i, j) \bar{R}_i(s) + \sum_{i=1}^4 \mathbf{H}_2(i, j) \bar{F}_i(s) + \frac{1}{M} \left[ \sum_{i=1}^2 \bar{R}_i(s) + \sum_{i=1}^4 \bar{F}_i(s) \right] . \tag{69}$$

The rotation matrix, say  $\mathcal{R}_{\text{wire}}^{(j)}$ , from the GRF to the preferred frame of wire #  $j$  is given by,

$$\mathcal{R}_{\text{wire}}^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi & \cos \phi \end{pmatrix} , \tag{70}$$

$$\mathcal{R}_{\text{wire}}^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & \sin \phi \\ 0 & -\sin \phi & \cos \phi \end{pmatrix} , \tag{71}$$

$$\sin \phi = \frac{d-h}{2l} . \tag{72}$$

Let the preferred frame of the  $j^{\text{th}}$  wire be denoted by  $PW_j$ .

The acceleration of the  $j^{\text{th}}$  wire-mirror attachment point in terms of the forces acting on the wire can be obtained from Eq. (49), (50) and Eq. (53). The acceleration components of the  $j^{\text{th}}$  in the GRF are,

$$s^2 \mathcal{R}_{\text{wire}}^{(j)T} \begin{pmatrix} -\frac{a_x^2}{T} W_x(l, 0, s) [\bar{G}_j]_{PW_{j,1}} + \frac{a_x^2}{T} W_x(l, l, s) [-\bar{R}_j]_{PW_{j,1}} \\ -\frac{a_y^2}{T} W_y(l, 0, s) [\bar{G}_j]_{PW_{j,2}} + \frac{a_y^2}{T} W_y(l, l, s) [-\bar{R}_j]_{PW_{j,2}} \\ -\frac{a_z^2}{Y} W_z(l, 0, s) [\bar{G}_j]_{PW_{j,3}} + \frac{a_z^2}{Y} W_z(l, l, s) [-\bar{R}_j]_{PW_{j,3}} \end{pmatrix} .$$

In the above expression we have used the fact that the force on the wire because of the mirror is opposite in direction to  $\bar{R}_j$  which was the force acting *on the mirror*. The equilibrium value of  $T$  is,

$$T = \frac{Mg}{\cos \phi}. \quad (73)$$

We can re-express the acceleration as,

$$s^2 \mathcal{R}_{\text{wire}}^{(j)T} \left[ \mathbf{W}_0 \mathcal{R}_{\text{wire}}^{(j)} \bar{G}_j + \mathbf{W}_1 \mathcal{R}_{\text{wire}}^{(j)} (-\bar{R}_j) \right],$$

$$\mathbf{W}_0 = \text{diag} \left[ -\frac{a_x^2}{T} W_x(l, 0, s), -\frac{a_y^2}{T} W_y(l, 0, s), -\frac{a_z^2}{Y} W_z(l, 0, s) \right]; \quad \mathbf{W}_1 = \text{diag} \left[ \frac{a_x^2}{T} W_x(l, l, s), \frac{a_y^2}{T} W_y(l, l, s), \frac{a_z^2}{Y} W_z(l, l, s) \right]. \quad (74)$$

Let  $\mathcal{R}_{\text{wire}}^{(j)T} \mathbf{W}_k \mathcal{R}_{\text{wire}}^{(j)}$  be denoted by  $\widetilde{\mathbf{W}}_k^{(j)}$ , for  $k = 0, 1$ . Then the acceleration reduces to,

$$s^2 \left[ \widetilde{\mathbf{W}}_0^{(j)} \bar{G}_j + \widetilde{\mathbf{W}}_1^{(j)} (-\bar{R}_j) \right], \quad (75)$$

where the explicit expressions for  $\widetilde{\mathbf{W}}_0^{(j)}$  and  $\widetilde{\mathbf{W}}_1^{(j)}$  are given in Appendix B.

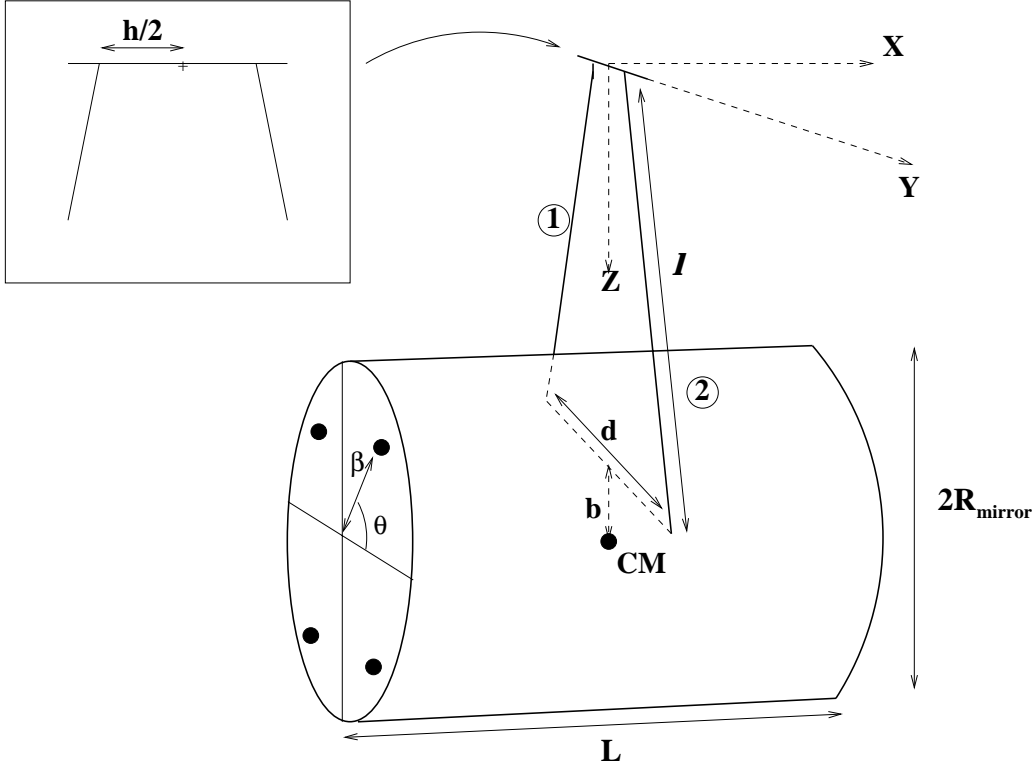


FIG. 4. A mirror suspension model. The mass of the mirror is  $M$  and the centre of mass (CM) lies directly on the  $Z$  axis. The length,  $l$ , of each wire is the length at mechanical equilibrium. The black circles on one of the faces show the location of control forces. Each location is at an angle  $\theta$  to the horizontal and at a distance  $\beta$  from the centre of the face. The control forces are assumed to be acting along the  $X$  axis.

We have to now equate the acceleration components given by Eq. (69) and Eq. (75). We get,

$$\sum_{i=1}^2 \mathbf{H}_1(i, 1) \bar{R}_i(s) + \sum_{i=1}^4 \mathbf{H}_2(i, 1) \bar{F}_i(s) + \frac{1}{M} \left[ \sum_{i=1}^2 \bar{R}_i(s) + \sum_{i=1}^4 \bar{F}_i(s) \right] = s^2 \left[ \widetilde{\mathbf{W}}_0^{(1)} \bar{G}_1(s) + \widetilde{\mathbf{W}}_1^{(1)} (-\bar{R}_1(s)) \right], \quad (76)$$

$$\sum_{i=1}^2 \mathbf{H}_1(i, 2) \bar{R}_i(s) + \sum_{i=1}^4 \mathbf{H}_2(i, 2) \bar{F}_i(s) + \frac{1}{M} \left[ \sum_{i=1}^2 \bar{R}_i(s) + \sum_{i=1}^4 \bar{F}_i(s) \right] = s^2 \left[ \widetilde{\mathbf{W}}_0^{(2)} \bar{G}_2(s) + \widetilde{\mathbf{W}}_1^{(2)} (-\bar{R}_2(s)) \right]. \quad (77)$$

Rewriting the above expressions, we get

$$\left[ \mathbf{H}_1(1, 1) + s^2 \widetilde{\mathbf{W}}_1^{(1)} + \frac{1}{M} \right] \bar{R}_1(s) + \left[ \mathbf{H}_1(2, 1) + \frac{1}{M} \right] \bar{R}_2(s) = s^2 \widetilde{\mathbf{W}}_0^{(1)} \bar{G}_1(s) - \sum_{i=1}^4 \left( \mathbf{H}_2(i, 1) + \frac{1}{M} \right) \bar{F}_i(s) \quad (78)$$

$$\left[ \mathbf{H}_1(1, 2) + \frac{1}{M} \right] \bar{R}_1(s) + \left[ \mathbf{H}_1(2, 2) + \frac{1}{M} + s^2 \widetilde{\mathbf{W}}_1^{(2)} \right] \bar{R}_2(s) = s^2 \widetilde{\mathbf{W}}_0^{(2)} \bar{G}_2(s) - \sum_{i=1}^4 \left( \mathbf{H}_2(i, 2) + \frac{1}{M} \right) \bar{F}_i(s) . \quad (79)$$

Finally, we can write the above expressions compactly as,

$$\mathbf{T} \mathbf{R} = \mathbf{F} , \quad (80)$$

where

$$\mathbf{R} = \begin{pmatrix} \bar{R}_1(s) \\ \bar{R}_2(s) \end{pmatrix} , \quad (81)$$

and the explicit expressions for  $\mathbf{T}$  and  $\mathbf{F}$  are given in Appendix B.

## APPENDIX A: EULER'S EQUATIONS

The angular motion of a rigid body can be obtained in terms of Euler's equations. We present a brief derivation of these equations now. In the space frame  $S$  attached to the centre of mass, the equation of motion is,

$$\frac{d\bar{\mathbf{L}}}{dt} = \bar{\mathbf{N}} , \quad (A1)$$

where  $\bar{\mathbf{L}}$  is the body's angular momentum and  $\bar{\mathbf{N}}$  is the total external torque as measured by an observer at rest in  $S$ . Let  $R_{b2s}$  be the rotation matrix from the body frame  $B$  to the  $S$ . Then,  $R_{s2b} = R_{b2s}^{-1} = R_{b2s}^T$ .

Eq. (A1) can be reexpressed in the body frame as,

$$\frac{d}{dt} (R_{b2s} [\bar{\mathbf{L}}]_B) = R_{b2s} [\bar{\mathbf{N}}]_B , \quad (A2)$$

where the subscript ' $B$ ' indicates the vector components in the body frame. We know that, for any vector  $\bar{\mathbf{X}}$ ,

$$\frac{dR_{b2s}}{dt} [\bar{\mathbf{X}}]_B = \bar{\boldsymbol{\Omega}} \times (R_{b2s} [\bar{\mathbf{X}}]_B) , \quad (A3)$$

where  $\bar{\boldsymbol{\Omega}}$  is the angular velocity in the space frame. Applying this to Eq. (A2), we get

$$\begin{aligned} (R_{b2s} [\bar{\boldsymbol{\Omega}}]_B) \times (R_{b2s} [\bar{\mathbf{L}}]_B) + R_{b2s} \left[ \dot{\bar{\mathbf{L}}} \right]_B &= R_{b2s} [\bar{\mathbf{N}}]_B , \\ [\bar{\boldsymbol{\Omega}}]_B \times [\bar{\mathbf{L}}]_B + \left[ \dot{\bar{\mathbf{L}}} \right]_B &= [\bar{\mathbf{N}}]_B . \end{aligned} \quad (A4)$$

Now,

$$\begin{aligned} \left[ \dot{\bar{\mathbf{L}}} \right]_B &= \sum_i m_i [\bar{\mathbf{r}}_i]_B \times ([\bar{\boldsymbol{\Omega}}]_B \times [\bar{\mathbf{r}}_i]_B) , \\ &= \sum_i m_i ([\bar{\mathbf{r}}_i]_B^2 [\bar{\boldsymbol{\Omega}}]_B - (\bar{\mathbf{r}}_i \cdot \bar{\boldsymbol{\Omega}}) [\bar{\mathbf{r}}_i]_B) , \\ &= \mathbf{I} [\bar{\boldsymbol{\Omega}}]_B , \end{aligned} \quad (A5)$$

where  $\mathbf{I}$  is the inertia tensor which is a constant in frame  $B$ ,

$$\mathbf{I}_{mn} = \sum_i m_i (\bar{\mathbf{r}}_i^2 \delta_{mn} - [\bar{\mathbf{r}}_i]_{B,m} [\bar{\mathbf{r}}_i]_{B,n}) , \quad (A6)$$

where the subscript ' $B, m$ ' denotes the  $m^{\text{th}}$  body frame component. For simplicity in the following, we will assume that the  $B$  frame axes are oriented along the principal axes of the body. By the

definition of principal axes, this implies that the inertia tensor is diagonal,  $\mathbf{I} = \text{diag}(I_1, I_2, I_3)$ . Substituting Eq. (A5) in Eq. (A4), we get

$$[\overline{\boldsymbol{\Omega}}]_B \times (\mathbf{I} [\overline{\boldsymbol{\Omega}}]_B) + \mathbf{I} [\dot{\overline{\boldsymbol{\Omega}}}]_B = [\overline{\mathbf{N}}]_B \quad (\text{A7})$$

or,

$$I_1 \dot{\Omega}_1 - (I_2 - I_3) \Omega_2 \Omega_3 = N_1, \quad (\text{A8})$$

$$I_2 \dot{\Omega}_2 - (I_3 - I_1) \Omega_3 \Omega_1 = N_2, \quad (\text{A9})$$

$$I_3 \dot{\Omega}_3 - (I_1 - I_2) \Omega_1 \Omega_2 = N_3, \quad (\text{A10})$$

where we have dropped the subscript ‘ $B$ ’ for convenience.

In order to get the equations motion for the angles, we must express  $\overline{\boldsymbol{\Omega}}$  in term of the rotational angles. It can be shown that if one chooses these angles to be the Euler angles, linearisation of Eq. (A8)–Eq. (A10) yields a non-dynamical equation of motion (i.e., the lowest order terms do not have the second derivatives of the angles). Thus one has to choose a different set of angles. Euler angles are given in Fig. (5) while the new set of angles called Kardan angles is given in Fig. (6).

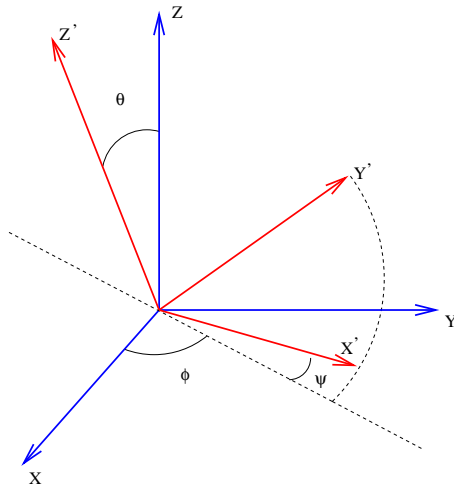


FIG. 5. The Euler angles  $(\theta, \phi, \psi)$ . The red axes (primed) are the body axes while the space axes (unprimed) are in blue. The angle  $\psi$  lies in the  $X'Y'$  plane.

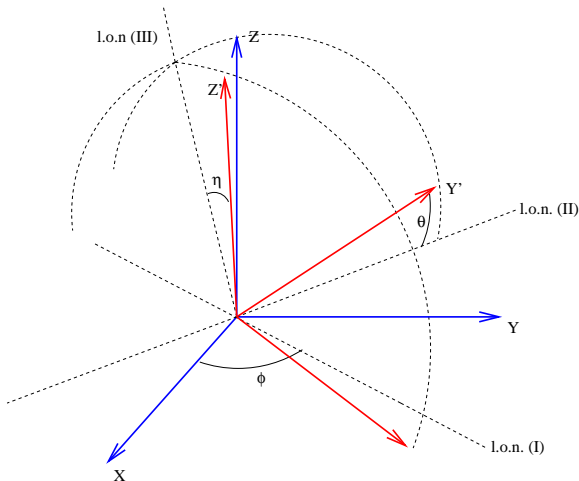


FIG. 6. The Kardan angles  $(\theta, \phi, \eta)$ . The red axes are the body axes while the space axes are in blue. The first rotation is around  $Z$  by angle  $\phi$  giving the line of nodes (l.o.n.) **l.o.n. (I)**. The second rotation is around the new  $X$  axis by an angle  $\theta$ . (as measured from **l.o.n. (II)**.) The third rotation is around the new  $Y$  axis,  $Y'$ , by an angle  $\eta$ . (measured from **l.o.n. (III)**.)

We can obtain  $\overline{\Omega}$  from the definition given in Eq. (A3). In terms of Euler angles,  $R_{b2s}$  is given in [3]. For Kardan angles,

$$R_{b2s} = [\mathbf{B} \mathbf{C} \mathbf{D}]^T, \quad \mathbf{B} = \begin{pmatrix} \cos \eta & 0 & -\sin \eta \\ 0 & 1 & 0 \\ \sin \eta & 0 & \cos \eta \end{pmatrix}, \quad (\text{A11})$$

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{pmatrix}, \quad (\text{A12})$$

$$\mathbf{D} = \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (\text{A13})$$

To first order in  $\theta$ ,  $\phi$  and  $\eta$ ,  $R_{b2s}$  is obtained as

$$R_{b2s} = \begin{pmatrix} 1 & -\phi & \eta \\ \phi & 1 & -\theta \\ -\eta & \theta & 1 \end{pmatrix}. \quad (\text{A14})$$

Finally  $\dot{R}_{b2s} R_{b2s}^T$  gives the components of  $\overline{\Omega}$ ,

$$\begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\dot{\phi} & \dot{\eta} \\ \dot{\phi} & 0 & -\dot{\theta} \\ -\dot{\eta} & \dot{\theta} & 0 \end{pmatrix} \quad (\text{A15})$$

To lowest order, Eq. (A8)–Eq. (A10) then become,

$$I_1 \ddot{\theta} = [\overline{N}]_{B,1}, \quad (\text{A16})$$

$$I_2 \ddot{\eta} = [\overline{N}]_{B,2}, \quad (\text{A17})$$

$$I_3 \ddot{\phi} = [\overline{N}]_{B,3}, \quad (\text{A18})$$

## APPENDIX B: SOME QUANTITIES REQUIRED FOR THE MIRROR SUSPENSION MODEL

The components, in frame  $B$ , of the points at which the wires are attached to the mirror are :

$$\overline{p}_1 = (0, -d/2, -b), \quad (\text{B1})$$

$$\overline{p}_2 = (0, d/2, -b), \quad (\text{B2})$$

for wire # 1 and #2 respectively. The  $B$  frame components for the points of application of control forces are :

$$\overline{P}_1 = (-L/2, \beta \cos \theta, -\beta \sin \theta), \quad (\text{B3})$$

$$\overline{P}_2 = (-L/2, -\beta \cos \theta, -\beta \sin \theta), \quad (\text{B4})$$

$$\overline{P}_3 = (-L/2, -\beta \cos \theta, \beta \sin \theta), \quad (\text{B5})$$

$$\overline{P}_4 = (-L/2, \beta \cos \theta, \beta \sin \theta). \quad (\text{B6})$$

We can now compute the matrices  $\mathbf{H}_1(i, j)$  and  $\mathbf{H}_2(i, j)$ .

$$\mathbf{H}_1(1, 1) = \begin{pmatrix} \frac{b^2}{I_c} + \frac{d^2}{4I_c} & 0 & 0 \\ 0 & \frac{b^2}{I_1} & \frac{-(b d)}{2I_1} \\ 0 & \frac{-(b d)}{2I_1} & \frac{d^2}{4I_1} \end{pmatrix}, \quad \mathbf{H}_1(2, 1) = \begin{pmatrix} \frac{b^2}{I_c} - \frac{d^2}{4I_c} & 0 & 0 \\ 0 & \frac{b^2}{I_1} & \frac{b d}{2I_1} \\ 0 & \frac{-(b d)}{2I_1} & \frac{d^2}{4I_1} \end{pmatrix}$$

$$\mathbf{H}_1(1, 2) = \begin{pmatrix} \frac{b^2}{I_c} - \frac{d^2}{4I_c} & 0 & 0 \\ 0 & \frac{b^2}{I_1} & \frac{-(b d)}{2I_1} \\ 0 & \frac{b d}{2I_1} & \frac{-d^2}{4I_1} \end{pmatrix}, \quad \mathbf{H}_1(2, 2) = \begin{pmatrix} \frac{b^2}{I_c} + \frac{d^2}{4I_c} & 0 & 0 \\ 0 & \frac{b^2}{I_1} & \frac{b d}{2I_1} \\ 0 & \frac{b d}{2I_1} & \frac{d^2}{4I_1} \end{pmatrix}$$

$$\begin{aligned}
\mathbf{H}_2(1,1) &= \begin{pmatrix} \frac{-(d\beta \cos \theta)}{2I_c} + \frac{b\beta \sin \theta}{I_c} & \frac{-(dL)}{4I_c} & \frac{-(bL)}{2I_c} \\ 0 & \frac{b\beta \sin \theta}{I_1} & \frac{b\beta \cos \theta}{I_1} \\ 0 & \frac{-(d\beta \sin \theta)}{2I_1} & \frac{-(d\beta \cos \theta)}{2I_1} \end{pmatrix}, \quad \mathbf{H}_2(2,1) = \begin{pmatrix} \frac{d\beta \cos \theta}{2I_c} + \frac{b\beta \sin \theta}{I_c} & \frac{-(dL)}{4I_c} & \frac{-(bL)}{2I_c} \\ 0 & \frac{b\beta \sin \theta}{I_1} & \frac{-b\beta \cos \theta}{I_1} \\ 0 & \frac{-(d\beta \sin \theta)}{2I_1} & \frac{d\beta \cos \theta}{2I_1} \end{pmatrix} \\
\mathbf{H}_2(3,1) &= \begin{pmatrix} \frac{d\beta \cos \theta}{2I_c} - \frac{b\beta \sin \theta}{I_c} & \frac{-(dL)}{4I_c} & \frac{-(bL)}{2I_c} \\ 0 & \frac{-b\beta \sin \theta}{I_1} & \frac{-b\beta \cos \theta}{I_1} \\ 0 & \frac{d\beta \sin \theta}{2I_1} & \frac{d\beta \cos \theta}{2I_1} \end{pmatrix}, \quad \mathbf{H}_2(4,1) = \begin{pmatrix} \frac{-(d\beta \cos \theta)}{2I_c} - \frac{b\beta \sin \theta}{I_c} & \frac{-(dL)}{4I_c} & \frac{-(bL)}{2I_c} \\ 0 & \frac{-b\beta \sin \theta}{I_1} & \frac{b\beta \cos \theta}{I_1} \\ 0 & \frac{d\beta \sin \theta}{2I_1} & \frac{-(d\beta \cos \theta)}{2I_1} \end{pmatrix} \\
\mathbf{H}_2(1,2) &= \begin{pmatrix} \frac{d\beta \cos \theta}{2I_c} + \frac{b\beta \sin \theta}{I_c} & \frac{dL}{4I_c} & \frac{-(bL)}{2I_c} \\ 0 & \frac{b\beta \sin \theta}{I_1} & \frac{b\beta \cos \theta}{I_1} \\ 0 & \frac{d\beta \sin \theta}{2I_1} & \frac{d\beta \cos \theta}{2I_1} \end{pmatrix}, \quad \mathbf{H}_2(2,2) = \begin{pmatrix} \frac{-(d\beta \cos \theta)}{2I_c} + \frac{b\beta \sin \theta}{I_c} & \frac{dL}{4I_c} & \frac{-(bL)}{2I_c} \\ 0 & \frac{b\beta \sin \theta}{I_1} & \frac{-b\beta \cos \theta}{I_1} \\ 0 & \frac{d\beta \sin \theta}{2I_1} & \frac{-(d\beta \cos \theta)}{2I_1} \end{pmatrix} \\
\mathbf{H}_2(3,2) &= \begin{pmatrix} \frac{-(d\beta \cos \theta)}{2I_c} - \frac{b\beta \sin \theta}{I_c} & \frac{dL}{4I_c} & \frac{-(bL)}{2I_c} \\ 0 & \frac{-b\beta \sin \theta}{I_1} & \frac{-b\beta \cos \theta}{I_1} \\ 0 & \frac{-(d\beta \sin \theta)}{2I_1} & \frac{-(d\beta \cos \theta)}{2I_1} \end{pmatrix}, \quad \mathbf{H}_2(4,2) = \begin{pmatrix} \frac{d\beta \cos \theta}{2I_c} - \frac{b\beta \sin \theta}{I_c} & \frac{dL}{4I_c} & \frac{-(bL)}{2I_c} \\ 0 & \frac{-b\beta \sin \theta}{I_1} & \frac{b\beta \cos \theta}{I_1} \\ 0 & \frac{-(d\beta \sin \theta)}{2I_1} & \frac{d\beta \cos \theta}{2I_1} \end{pmatrix} \quad (\text{B7})
\end{aligned}$$

The matrices  $\widetilde{\mathbf{W}}_k^{(j)}$  are as follows.

$$\begin{aligned}
\widetilde{\mathbf{W}}_0^{(1)} &= \begin{pmatrix} -\frac{a_x^2 W_{x0}}{T} & 0 & 0 \\ 0 & \frac{\cos(\varphi)^2 a_y^2 W_{y0} - \sin(\varphi)^2 a_z^2 W_{z0}}{T} & \frac{\cos(\varphi) \sin(\varphi) (Y a_y^2 W_{y0} - T a_z^2 W_{z0})}{TY} \\ 0 & \frac{\cos(\varphi) \sin(\varphi) (Y a_y^2 W_{y0} - T a_z^2 W_{z0})}{TY} & \frac{\sin(\varphi)^2 a_y^2 W_{y0} - \cos(\varphi)^2 a_z^2 W_{z0}}{Y} \end{pmatrix}, \\
\widetilde{\mathbf{W}}_0^{(2)} &= \begin{pmatrix} -\frac{a_x^2 W_{x0}}{T} & 0 & 0 \\ 0 & \frac{\cos(\varphi)^2 a_y^2 W_{y0} - \sin(\varphi)^2 a_z^2 W_{z0}}{T} & \frac{\cos(\varphi) \sin(\varphi) (-Y a_y^2 W_{y0} + T a_z^2 W_{z0})}{TY} \\ 0 & \frac{\cos(\varphi) \sin(\varphi) (-Y a_y^2 W_{y0} + T a_z^2 W_{z0})}{TY} & \frac{-\sin(\varphi)^2 a_y^2 W_{y0} - \cos(\varphi)^2 a_z^2 W_{z0}}{Y} \end{pmatrix}, \\
\widetilde{\mathbf{W}}_1^{(1)} &= \begin{pmatrix} \frac{a_x^2 W_{xl}}{T} & 0 & 0 \\ 0 & \frac{\cos(\varphi)^2 a_y^2 W_{yl} + \sin(\varphi)^2 a_z^2 W_{zl}}{T} & \frac{\cos(\varphi) \sin(\varphi) (-Y a_y^2 W_{yl} + T a_z^2 W_{zl})}{TY} \\ 0 & \frac{\cos(\varphi) \sin(\varphi) (-Y a_y^2 W_{yl} + T a_z^2 W_{zl})}{TY} & \frac{\sin(\varphi)^2 a_y^2 W_{yl} + \cos(\varphi)^2 a_z^2 W_{zl}}{Y} \end{pmatrix}, \\
\widetilde{\mathbf{W}}_1^{(2)} &= \begin{pmatrix} \frac{a_x^2 W_{xl}}{T} & 0 & 0 \\ 0 & \frac{\cos(\varphi)^2 a_y^2 W_{yl} + \sin(\varphi)^2 a_z^2 W_{zl}}{T} & \frac{\cos(\varphi) \sin(\varphi) (Y a_y^2 W_{yl} - T a_z^2 W_{zl})}{TY} \\ 0 & \frac{\cos(\varphi) \sin(\varphi) (Y a_y^2 W_{yl} - T a_z^2 W_{zl})}{TY} & \frac{\sin(\varphi)^2 a_y^2 W_{yl} + \cos(\varphi)^2 a_z^2 W_{zl}}{Y} \end{pmatrix}. \quad (\text{B8})
\end{aligned}$$

where  $W_{x0} \equiv W_x(l, 0, s)$ ,  $W_{y0} \equiv W_y(l, 0, s)$ ,  $W_{z0} \equiv W_z(l, 0, s)$  and  $W_{xl} \equiv W_x(l, l, s)$ ,  $W_{yl} \equiv W_y(l, l, s)$ ,  $W_{zl} \equiv W_z(l, l, s)$ .

The matrix  $\mathbf{T}$  is presented below. Here,  $W_x \equiv W_x(l, l, s)$ ,  $W_y \equiv W_y(l, l, s)$  and  $W_z \equiv W_z(l, l, s)$ . The expression for  $\phi$  is provided in Eq. (??).

$$\mathbf{T} = \begin{pmatrix} \mathbf{T}_1 & \mathbf{T}_2 \\ \mathbf{T}_3 & \mathbf{T}_4 \end{pmatrix}, \quad (\text{B9})$$

$$\mathbf{T}_1 = \begin{pmatrix} \frac{b^2}{I_c} + \frac{d^2}{4I_c} + \frac{s^2 a_x^2 W_x}{T} + \frac{1}{M} & 0 & 0 \\ 0 & \frac{b^2}{I_1} + s^2 \left[ \frac{a_y^2 W_y \cos^2 \phi}{T} + \frac{a_z^2 W_z \sin^2 \phi}{Y} \right] + \frac{1}{M} & -\frac{bd}{2I_1} + \frac{s^2 (-Y a_y^2 W_y + T a_z^2 W_z) \cos \phi \sin \phi}{TY} \\ 0 & \frac{-bd}{2I_1} + \frac{s^2 (-Y a_y^2 W_y + T a_z^2 W_z) \cos \phi \sin \phi}{TY} & \frac{d^2}{4I_1} + \frac{1}{M} + s^2 \left[ \frac{a_y^2 W_y \sin^2 \phi}{T} + \frac{a_z^2 W_z \cos^2 \phi}{Y} \right] \end{pmatrix}, \quad (\text{B10})$$

$$\mathbf{T}_2 = \begin{pmatrix} \frac{b^2}{I_c} - \frac{d^2}{4I_c} + \frac{1}{M} & 0 & 0 \\ 0 & \frac{b^2}{I_1} + \frac{1}{M} & \frac{bd}{2I_1} \\ 0 & \frac{-bd}{2I_1} & \frac{-d^2}{4I_1} + \frac{1}{M} \end{pmatrix}, \quad (\text{B11})$$

$$\mathbf{T}_3 = \begin{pmatrix} \frac{b^2}{I_c} - \frac{d^2}{4I_c} + \frac{1}{M} & 0 & 0 \\ 0 & \frac{b^2}{I_1} + \frac{1}{M} & \frac{-bd}{2I_1} \\ 0 & \frac{bd}{2I_1} & \frac{1}{M} - \frac{d^2}{4I_1} \end{pmatrix}, \quad (\text{B12})$$

$$\mathbf{T}_4 = \begin{pmatrix} \frac{b^2}{I_c} + \frac{d^2}{4I_c} + s^2 \frac{a_x^2}{T} W_x + \frac{1}{M} & 0 & 0 \\ 0 & \frac{b^2}{I_1} + s^2 \left[ \frac{a_y^2 W_y \cos^2 \phi}{T} + \frac{a_z^2 W_z \sin^2 \phi}{Y} \right] & \frac{bd}{2I_1} + \frac{s^2 (Y a_y^2 W_y - T a_z^2 W_z) \cos \phi \sin \phi}{TY} \\ 0 & \frac{bd}{2I_1} + \frac{s^2 (Y a_y^2 W_y - T a_z^2 W_z) \cos \phi \sin \phi}{TY} & \frac{1}{M} + \frac{d^2}{4I_1} + s^2 \left[ \frac{a_y^2 W_y \sin^2 \phi}{T} + \frac{a_z^2 W_z \cos^2 \phi}{Y} \right] \end{pmatrix}. \quad (\text{B13})$$

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- [1] Anatoliy G. Butkovskiy, *Green's Functions and transfer functions handbook*, 1982 (John Wiley, New York).
  - [2] Gabriela I. Gonzalez, Peter R. Saulson, *J. Acoust. Soc. Am.* **96** (1), 207 (1994).
  - [3] Herbert Goldstein, *Classical Mechanics*, Seventh edition, 1965 (Addison-Wesley, Massachusetts).