

LASER INTERFEROMETER GRAVITATIONAL WAVE OBSERVATORY
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CALIFORNIA INSTITUTE OF TECHNOLOGY
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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**Time Domain Modal Model
in End-to-End simulation package**

Biplab Bhawal, Matt Evans,
Malik Rakhmanov and Hiro Yamamoto

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California Institute of Technology
LIGO Project - MS 51-33
Pasadena CA 91125
Phone (626) 395-2129
Fax (626) 304-9834
E-mail: info@ligo.caltech.edu

Massachusetts Institute of Technology
LIGO Project - MS 20B-145
Cambridge, MA 01239
Phone (617) 253-4824
Fax (617) 253-7014
E-mail: info@ligo.mit.edu

WWW: <http://www.ligo.caltech.edu/>

1 ABSTRACT

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2 KEYWORDS

xxx

3 DEFINITION OF WORDS

mode support

4 MODAL MODEL

4.1. General Formula

$$E(x, y, z) = \sum a_{mn} \cdot U_{mn}(x, y, z) \quad (1)$$

$$U_{mn}(x, y, z) = u_m(x, z) \cdot u_n(y, z) \cdot \exp(-ik \cdot z) \quad (2)$$

$$u_m(x, z) = \left(\frac{2}{\pi}\right)^{\frac{1}{4}} \cdot \left(\frac{1}{2^m m! w(z)}\right)^{\frac{1}{2}} \cdot \exp\left(i\left(m + \frac{1}{2}\right)\eta_{00}(z)\right) \quad (3)$$

$$H_m\left(\frac{\sqrt{2}x}{w(z)}\right) \cdot \exp\left(-x^2\left(\frac{1}{w(z)^2} + \frac{ik}{2R(z)}\right)\right)$$

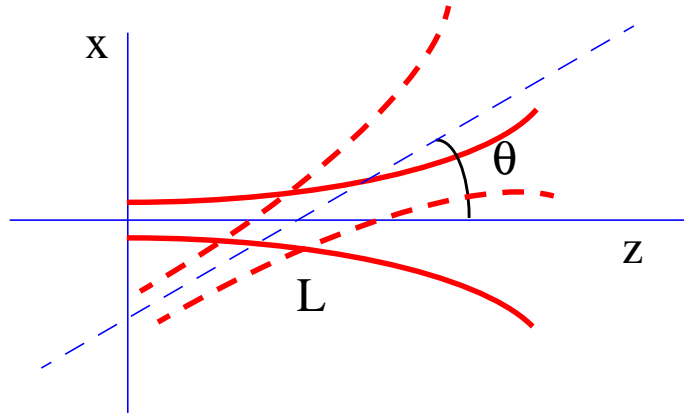
$$\begin{aligned} Op[E(x, y, z)] &= \sum_{mn} Op[a_{mn} \cdot U_{mn}(x, y, z)] \\ &= \sum_{mn} a_{mn} \cdot Op[u_m(x, z)]Op[u_n(y, z)] \\ &= \sum_{mn} a'_{mn} \cdot U_{mn}(x, y, z) \end{aligned} \quad (4)$$

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$$\begin{aligned}
a'_{mn} &= \int dx dy U_{mn}^*(x, y, z) Op[E(z, y, z)] \\
&= \sum_{m'n'} a_{m'n'} \int dx u_m^*(x, z) Op[u_{m'}(x, z)] \int dy u_n^*(y, z) Op[u_{n'}(y, z)] \\
&= \sum_{m'n'} a_{m'n'} \cdot M_{mm'}^{Op} \cdot M_{nn'}^{Op}
\end{aligned} \tag{5}$$

4.2. Rotation Operator

4.2.1. Rotate around (0,L) by θ



$$Op[E(x, y, z)] = E(x \cdot \cos \theta - (z - L) \cdot \sin \theta, y, L + x \cdot \sin \theta + (z - L) \cdot \cos \theta) \tag{6}$$

4.2.2. Mode decomposition matrix

$$\begin{aligned}
M_{mm'}^{Rot} &= \int dx u_m^*(x, L) Rot[u_{m'}(x, L)] \\
&= i^{-|\Delta m|} \frac{\sqrt{m! m'!}}{\bar{m}! \cdot |\Delta m|!} e^{i \Delta m \cdot \eta_{00}} e^{-\frac{\hat{\theta}^2}{2}} \hat{\theta}^{|\Delta m|} g(m, m', \hat{\theta})
\end{aligned} \tag{7}$$

$$\Delta m = m' - m$$

$$\bar{m} = \min(m, m')$$

$$\hat{\theta} = \frac{w(L)}{w_0} \cdot \frac{\theta}{\Theta} = \sqrt{1 + \left(\frac{L}{z_0}\right)^2} \cdot \frac{\theta}{w_0/z_0} = \frac{\sqrt{z_0^2 + L^2}}{w_0} \cdot \theta \tag{8}$$

$$g(m, m', x) = \bar{m}! \cdot |\Delta m|! \cdot \sum_{r=0}^{\bar{m}} \frac{(-1)^r}{r!(\bar{m}-r)! (|\Delta m|+r)!} x^{2r} \quad (9)$$

$$\Delta m = m' - m$$

$$\bar{m} = \min(m, m')$$

$$\eta_{00} = \text{atan}\left(\frac{L}{z_0}\right) = \frac{z_0 + iL}{\sqrt{z_0^2 + L^2}} \quad (10)$$

4.2.3. Lower order elements

Table 1: $M_{mm'}^{Rot}$, excluding $\exp\left(-\frac{\hat{\theta}^2}{2}\right) \cdot \exp(i(m' - m)\eta_{00})$

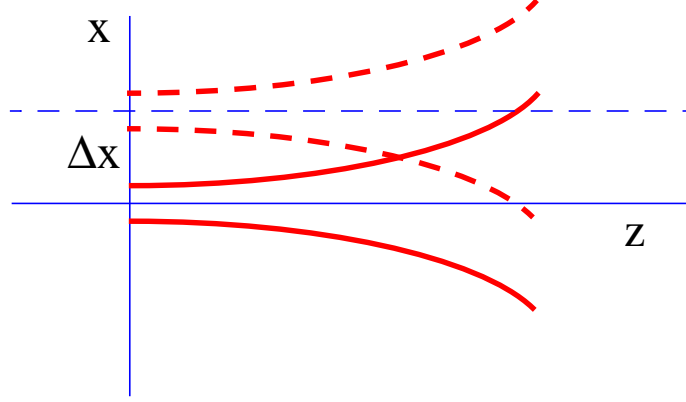
$m \backslash m'$	0	1	2	3
0	1	$-i\hat{\theta}$	$-\frac{\hat{\theta}^2}{\sqrt{2}}$	$i\frac{\hat{\theta}^3}{\sqrt{6}}$
1	$-i\hat{\theta}$	$1 - \hat{\theta}^2$	$-\sqrt{2}i\hat{\theta}\left(1 - \frac{\hat{\theta}^2}{2}\right)$	$-\sqrt{\frac{3}{2}}\hat{\theta}^2\left(1 - \frac{\hat{\theta}^2}{3}\right)$
2	$-\frac{\hat{\theta}^2}{\sqrt{2}}$	$-\sqrt{2}i\hat{\theta}\left(1 - \frac{\hat{\theta}^2}{2}\right)$	$1 - 2\hat{\theta}^2 + \frac{\hat{\theta}^4}{2}$	$-i\sqrt{3}\hat{\theta}\left(1 - \hat{\theta}^2 + \frac{\hat{\theta}^4}{6}\right)$
3	$i\frac{\hat{\theta}^3}{\sqrt{6}}$	$-\sqrt{\frac{3}{2}}\hat{\theta}^2\left(1 - \frac{\hat{\theta}^2}{3}\right)$	$-i\sqrt{3}\hat{\theta}\left(1 - \hat{\theta}^2 + \frac{\hat{\theta}^4}{6}\right)$	$1 - 3\hat{\theta}^2 + \frac{3\hat{\theta}^4}{2} - \frac{\hat{\theta}^6}{6}$

4.2.4. Sanity check

$$\sum_{m3} M_{m1m3}^{Rot}(-\hat{\theta}) \cdot M_{m3m2}^{Rot}(\hat{\theta}) = \delta_{m1m2} \quad (11)$$

4.3. Shift Operator

4.3.1. Shift perpendicular to z axis



$$Op[E(x, y, z)] = E(x - \Delta x, y, z) \quad (12)$$

4.3.2. Mode decomposition matrix

$$\begin{aligned} M_{mm'}^{shift} &= \int dx u_m^*(x, L) Shift[u_{m'}(x, L)] \\ &= (-1)^{m + \max(m, m')} \frac{\sqrt{m!m'}}{\bar{m}! \cdot |\Delta m|!} \cdot e^{-\frac{\hat{\Delta}^2}{2}} \hat{\Delta}^{|\Delta m|} g(m, m', \hat{\Delta}) \end{aligned} \quad (13)$$

$$\Delta m = m' - m$$

$$\bar{m} = \min(m, m')$$

$$\hat{\Delta} = \frac{\Delta x}{w_0} \quad (14)$$

4.3.3. Lower order elements

4.3.4. Sanity check

$$\sum_{m_3} M_{m_1 m_3}^{shift}(-\hat{\Delta}) \cdot M_{m_3 m_2}^{shift}(\hat{\Delta}) = \delta_{m_1 m_2} \quad (15)$$

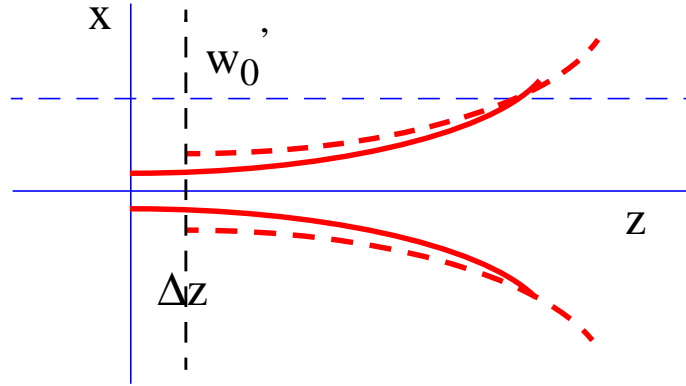
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Table 2: $M_{mm'}^{Shift}$ excluding $\exp\left(-\frac{\hat{\Delta}^2}{2}\right)$

$m \backslash m'$	0	1	2	3
0	1	$-\hat{\Delta}$	$\frac{\hat{\Delta}^2}{\sqrt{2}}$	$-\frac{\hat{\Delta}^3}{\sqrt{6}}$
1	$\hat{\Delta}$	$1-\hat{\Delta}^2$	$-\sqrt{2}\hat{\Delta}\left(1-\frac{\hat{\Delta}^2}{2}\right)$	$\sqrt{\frac{3}{2}}\hat{\Delta}^2\left(1-\frac{\hat{\Delta}^2}{3}\right)$
2	$\frac{\hat{\Delta}^2}{\sqrt{2}}$	$\sqrt{2}\hat{\Delta}\left(1-\frac{\hat{\Delta}^2}{2}\right)$	$1-2\hat{\Delta}^2+\frac{\hat{\Delta}^4}{2}$	$-\sqrt{3}\hat{\Delta}\left(1-\hat{\Delta}^2+\frac{\hat{\Delta}^4}{6}\right)$
3	$\frac{\hat{\Delta}^3}{\sqrt{6}}$	$\sqrt{\frac{3}{2}}\hat{\Delta}^2\left(1-\frac{\hat{\Delta}^2}{3}\right)$	$\sqrt{3}\hat{\Delta}\left(1-\hat{\Delta}^2+\frac{\hat{\Delta}^4}{6}\right)$	$1-3\hat{\Delta}^2+\frac{3\hat{\Delta}^4}{2}-\frac{\hat{\Delta}^6}{6}$

4.4. Base Change

4.4.1. Change of waist size and position



$$Op[E(x, y, z, w_0)] = E(x, y, z - \Delta z, w_0') \quad (16)$$

4.4.2. Mode decomposition matrix

Keep only 1st order of $\delta\hat{w}_0$ and $\delta\hat{z}$. Expression valid when

$$\delta\hat{w}_0, \delta\hat{z} < \sqrt{\frac{w_0}{z_0}} \quad (17)$$

$$\begin{aligned}
M_{mm'}^{shift} &= \int dx u_m^*(x, z) \text{Base}[u_{m'}(x, z)] \\
&= \exp[ik\Delta z] \times \left\{ \begin{array}{l} \frac{-\frac{1}{2} \exp(2i\eta_{00})}{\sqrt{(m+1)(m+2)}} \left(\delta\hat{w}_0 + i\frac{1}{2}\delta\hat{z} \right) \quad m' = m+2 \\ 1 - i\frac{1}{2} \left(m + \frac{1}{2} \right) \delta\hat{z} \quad m' = m \\ \frac{1}{2} \exp(-2i\eta_{00}) \left(\delta\hat{w}_0 - i\frac{1}{2}\delta\hat{z} \right) \quad m' = m-2 \end{array} \right\} \quad (18)
\end{aligned}$$

$$\delta\hat{w}_0 = \frac{w_0'}{w_0} - 1$$

$$\delta\hat{z} = \frac{\Delta z}{z_0}$$

$$\eta_{00} = \text{atan}\left(\frac{z}{z_0}\right) \quad (19)$$

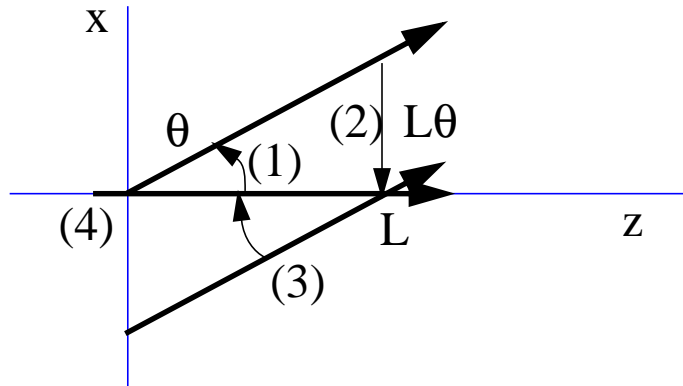
$$k\Delta z = \frac{2z_0}{w_0^2} \cdot \Delta z$$

4.4.3. Sanity check

$$\sum_{m3} M_{m1m3}^{Base}(-\delta\hat{z}, -\delta\hat{w}_0) \cdot M_{m3m2}^{Base}(\delta\hat{z}, \delta\hat{w}_0) = \delta_{m1m2} \quad (20)$$

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4.5. Another sanity check



- (1) Rotate at (0,0) by θ , with waist at (0,0)
- (2) Shift by $-L\theta$ toward $-x$ direction, with waist at $(-L\theta, 0)$
- (3) Rotate at (0,L) by $-\theta$, with waist at $(0, -L\theta^2/2)$
- (4) Move the waist position by $L\theta^2/2$ toward $+z$ direction to move waist to (0,0)

$$M^{Base}\left(\Delta z = \frac{L\theta^2}{2}\right)M^{Rot}(L, -\theta)M^{Shift}(-L\theta)M^{Rot}(0, \theta) = I \quad (21)$$

5 PHYSICS QUANTITIES

Table 3: Physics parameters of interest

<i>name</i>	<i>value</i>	<i>comment</i>
LIGO ITM curvature	14.18 km	
LIGO ETM curvature	7.4 km	
LIGO Recycling curv	14.9 km	
LIGO ITM Trans / Loss	3% / 50 ppm	
LIGO ETM Trans / Loss	< 20ppm / 50 ppm	
LIGO Recycling T/L	3% / 50 ppm	

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Table 4: field parameters

<i>name</i>	<i>expression</i>	<i>in COC (4k/2k)</i>	<i>in PSL/IOO</i>
waist size	w_0	3.51 / 3.13 cm	
Raylay length	$z_0 = \frac{\pi \cdot w_0^2}{\lambda} = \frac{k \cdot w_0^2}{2} = \frac{w_0}{\Theta}$ $= \sqrt{\frac{L(R_1 - L)(R_2 - L)(R_1 + R_2 - L)}{(R_1 + R_2 - 2L)^2}}$	3.63 / 2.89 km	
Distance to waist	$z_1 = \frac{L(R_2 - L)}{R_1 + R_2 - 2L}, z_2 = \frac{L(R_1 - L)}{R_1 + R_2 - 2L}$	1000, 3000 m / 614, 1386 m	
spot size	$w = w_0 \sqrt{1 + \left(\frac{z}{z_0}\right)^2}$	3.64, 4.55cm / 3.20, 3.47cm size @ITM,ETM	
curvature of phase front	$R(z) = \frac{z^2 + z_0^2}{z}$		
divergence angle	$\Theta = \frac{\lambda}{\pi \cdot w_0} = \frac{w_0}{z_0}$	9.7 / 10.8 μ r	
guoy phase	$\eta(z) = \text{atan}\left(\frac{z}{z_0}\right) = \text{angle}\left(\frac{z_0 + iz}{ z_0 + iz }\right)$	-15.4, 39.6 deg -12.0, 25.6 deg η @ITM/ETM	
wave number	$k = \frac{2\pi}{\lambda} = \frac{2}{\Theta \cdot w_0} = \frac{2z_0}{w_0^2}$	5.91e6~YAG 0.513 ~24.5MHz 0.618 ~29.5MHz	

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6 HARMIT GAUSSIAN FUNCTIONS

$H_m(x)$ is the Hermite polynomial of order m . The following relations are used repeatedly in the calculations which follow:

$$\int_{-\infty}^{\infty} U_m^\dagger(x, z) U_n(x, z) dx = \delta_{mn} \quad (22)$$

$$2xH_m(x) = H_{m+1}(x) + 2mH_{m-1}(x) \quad (23)$$

$$\frac{d}{dx}H_m(x) = 2mH_{m-1}(x) \quad (24)$$

$$\int_{-\infty}^{\infty} U_m^\dagger(x, 0) \frac{H_i(\sqrt{2}x/w_0)}{H_k(\sqrt{2}x/w_0)} U_k(x, 0) dx = \sqrt{\frac{2^i i!}{2^k k!}} \delta_{mi} \quad (25)$$

Eqn. (22) is the orthonormality condition; eqns. (23) and (24) are recursion relations to be used to derive Hermite polynomials of any order, beginning with $H_0(x) = 1$.

In two dimensions the Hermite-Gaussian modes are given by

$$U_{mn} = U_m(x, z)U_n(y, z)e^{-ikz} \quad (26)$$

The explicit forma of a few low order Hermite polynomials are :

$$H_0(x) = 1; H_1(x) = 2x; H_2(x) = -2 + 4x^2; H_3(x) = -12x + 8x^3; \quad (27)$$

A few examples of the distribution of the Hermite Gaussians are shown below.

The following formula is used in the modeling of the photo detector with simple shapes.

$$\int_0^{\infty} H_n(x)H_m(x) \exp(-x^2) dx = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{s=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^{r+s} \cdot \frac{n!m!2^{n+m-1}}{(2r)!!(2s)!!(n-2r)!(m-2s)!} \cdot \begin{cases} \left(\frac{n+m-1}{2} - r - s\right)! \cdot 2^{-r-s} & \text{for odd } n+m \\ \frac{(n+m-2r-2s-1)!!}{\sqrt{2}^{n+m}} \sqrt{\pi} & \text{for even } n+m \end{cases} \quad (28)$$

$$\text{Table 5: } HH[m, n] = \frac{1}{\sqrt{\pi}2^{m+n}m!n!} \int_0^{\infty} H_m(x)H_n(x)\exp(-x^2)$$

m \ n	0	1	2	3	4	5
0	$\frac{1}{2}$	$\frac{1}{\sqrt{2\pi}}$	0	$-\frac{1}{2\sqrt{3\pi}}$	0	$\frac{1}{4}\sqrt{\frac{3}{5\pi}}$
1	$\frac{1}{\sqrt{2\pi}}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{\pi}}$	0	$-\frac{1}{4\sqrt{3\pi}}$	0
2	0	$\frac{1}{2\sqrt{\pi}}$	$\frac{1}{2}$	$\frac{1}{2}\sqrt{\frac{3}{2\pi}}$	0	$-\frac{1}{4}\sqrt{\frac{5}{6\pi}}$
3	$-\frac{1}{2\sqrt{3\pi}}$	0	$\frac{1}{2}\sqrt{\frac{3}{2\pi}}$	$\frac{1}{2}$	$\frac{3}{4\sqrt{2\pi}}$	0
4	0	$-\frac{1}{4\sqrt{3\pi}}$	0	$\frac{3}{4\sqrt{2\pi}}$	$\frac{1}{2}$	$\frac{3}{8}\sqrt{\frac{5}{2\pi}}$
5	$\frac{1}{4}\sqrt{\frac{3}{5\pi}}$	0	$-\frac{1}{4}\sqrt{\frac{5}{6\pi}}$	0	$\frac{3}{8}\sqrt{\frac{5}{2\pi}}$	$\frac{1}{2}$

$$\int_{-\infty}^{\infty} H_n(x)H_m(x)\exp(-x^2) = \begin{cases} \sqrt{\pi}2^n n! & \text{for } n=m \\ 0 & \text{for } n \neq m \end{cases} \quad (29)$$

$$H_n(x) = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^r (2r-1)!! {}_n C_{2r} 2^{n-r} x^{n-2r} \quad (30)$$

$$H_n(x+y) = \sum_{r=0}^n H_{n-r}(x) 2^r {}_n C_r y^r \quad (31)$$

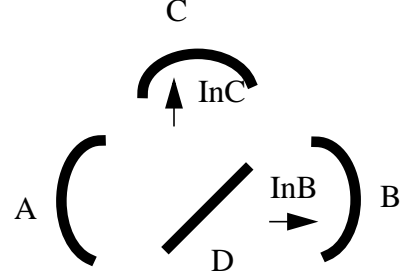
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7 THE RECYCLING SUMMATION CAVITY CODE

This recycling summation cavity module can be used for faster computation in simulating either a power-recycled Michelson interferometer or the recycling cavity of LIGO.

For one round-trip the field evolution equation can be written as a matrix equation

$$\overline{In}(t + \tau) = \overline{F} \times \overline{In}(\tilde{t}) + \overline{A}_{in} \quad (32)$$



Three vectors and one matrix (hereafter called the Finesse matrix, \overline{F}) in the above equation can be identified in the following equation:

$$\begin{bmatrix} InC(t + \tau) \\ InB(t + \tau) \end{bmatrix} = \begin{bmatrix} C2C & B2C \\ C2B & B2B \end{bmatrix} \begin{bmatrix} InC(t + (2\Delta)/c) \\ InB(t - (2\Delta)/c) \end{bmatrix} + \begin{bmatrix} A2C \\ A2B \end{bmatrix} \quad (33)$$

where

$$C2C = \Phi[DtoC]\hat{R}_D\Phi[AtoD]\hat{R}_A\Phi[DtoA]\hat{R}_D\Phi[CtoD]\hat{R}_C \quad (34)$$

$$B2C = \Phi[DtoC]\hat{R}_D\Phi[AtoD]\hat{R}_A\Phi[AtoD]T_D\Phi[BtoA]\hat{R}_B \quad (35)$$

$$B2B = \Phi[AtoB]\hat{T}_D\hat{R}_A\hat{T}_D\Phi[BtoA]\hat{R}_B \quad (36)$$

$$C2B = \hat{T}_D\Phi[AtoB]\hat{R}_A\Phi[DtoA]\hat{R}_D\Phi[CtoD]\hat{R}_C \quad (37)$$

$$A2C = A_0\hat{T}_A\Phi[AtoD]\hat{R}_D\Phi[DtoC] \quad (38)$$

$$A2B = A_0\hat{T}_A\Phi[AtoD]\hat{T}_D\Phi[DtoB] \quad (39)$$

where A_0 is the input laser amplitude; $\Phi[AtoD]$ etc. represent the total acquired phase (guoy + longitudinal) in traversing 'the distance between mirror A and Beam-splitter' etc. ; \hat{R}_A , \hat{T}_A etc. represent reflection or transmission operation from mirror A, etc. If average length of the recycling cavity is assumed to be l , the distance between the recycling mirror and mirror C is taken to be $l - \Delta$ and the distance between the recycling mirror and mirror B is taken to be $l + \Delta$; The round-trip-time τ is equal to $(2l)/c$.

The transmission operator, e.g. \hat{T}_A , includes multiplication by amplitude-transmittivity and changes in modal-basis of the beam (i.e., the lensing effect). The reflection operator, e.g. \hat{R}_A , includes multiplication by amplitude-reflectivity with appropriate sign (in E2E's convention, minus for coated side and plus for the side of dielectric) and a composite matrix representing

small perturbation in optics like rotation or shift of the mirror or mismatch in its radius of curvature with the incoming beam.

Ref.2 discusses a method of fast-simulation for a 2-mirror cavity as well as more complicated systems like coupled cavities and LIGO-type interferometer by making an analogy of the latter cases with a 2-mirror cavity. The summation formula (Eq.(92) of Ref.2) which allows a jump of N round-trip times can be rewritten here in a more convenient notation as:

$$\bar{I}n(t + N\tau) = \bar{F}^N \bar{E} \bar{I}n(\tilde{t}) + \frac{(\bar{U} - \bar{F}^N)}{(\bar{U} - \bar{F})} \bar{A}_{in} + \frac{[\bar{F} \bar{S} - 0.5(N^2 - 1)\bar{F}^N]}{(\bar{U} - \bar{F})} \bar{\xi} \bar{A}_{in} \quad (40)$$

where \bar{U} is the unity matrix, and

$$\bar{E} = \begin{bmatrix} \exp(0.5\xi_c N^2) & 0 \\ 0 & \exp(0.5\xi_b N^2) \end{bmatrix} \quad (41)$$

$$\bar{\xi} = \begin{bmatrix} \xi_c & 0 \\ 0 & \xi_b \end{bmatrix} \quad (42)$$

$$\xi_c = j \frac{4\pi W_{ca} \tau}{\lambda} \quad \xi_b = j \frac{4\pi W_{ba} \tau}{\lambda} \quad (43)$$

(where W_{ca} and W_{ba} are relative velocities of mirrors C and B wrt mirror A)

$$\bar{S} = \frac{[(N - 0.5)\bar{U} - 1.5\bar{F}^{N-1}]}{(\bar{U} - \bar{F})} + \frac{(\bar{F}^{N-1} - \bar{F})}{(\bar{U} - \bar{F})^2} \quad (44)$$

While reading the code (currently, Jan12-2000, in file smoptics.cc) the following points may be remembered:

- In each iteration the code needs to redo the “final” calculation using Eq.(40) whenever one of the following conditions changes: (i) any mirror is moved, (ii) alignment of any mirror is changed (iii) the size of the time-step is changed (which may happen while running “modeler_freq” which may choose a different time-step whenever the run switches over to a new frequency).
- If there’s no longitudinal motion in any mirror, the matrix $\bar{\xi}$ becomes a null matrix and the third term in Eq.(40) is zero.
- The code is written in a way such that the computationally expensive matrix calculations required for misalignment effects are not unnecessarily repeated. For example, if only the mirror is moving, the misalignment matrix operations are not repeated. To keep computationally expensive alignment calculations separate from calculations dependent on the longitudinal

motion, the quantities like, $C2B$ etc are calculated in two parts, one involving combinations of amplitude-reflectivities, longitudinal phase factors etc. and other involving products of alignment matrices at various mirrors and guoy phases. This allows putting appropriate tags so that if one or both aspects (logitudinal, alignment) donot change, the code does not unnecessarily repeat those calculations.

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APPENDIX 1 REFERENCE

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2. Bhawal, B., JOSA A, **105**, 120-143 (1998)

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