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Optical Cavity Time Constant Measurement with Two in Series Optical Resonators

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1 Introduction

A standard technique to measure the optical resonator time constant[1] is to record and fit the cavity response to a step input (blocking the input light). This technique is known to work remarkably well when the input light can be shut down faster than the optical cavity time constant. The faster the light is shut the better the cavity light decay will be approximated by the ideal case of a simple electric field ring-down.

This technical note describe a way to measure the time constant of a cavity when input light cannot be shut down fast enough, i.e. when the time constant of the decay input intensity is comparable to the cavity time constant. This is the case of the 40m Fabry-Perot cavities where the 12m mode cleaner where the two times constants differs by a about factor 5.

Technical limitations are essentially due to the ability of the numerical fitting procedure.

2 Fabry-Perot Transfer Function

Lets consider a Fabry-Perot cavity of length l and mirrors with transmission and reflection coefficients $t_1, t_{,2}$ and ir_1, ir_2 respectively. Supposing that the monochromatic incident field is $E_i(t)$, the field transmitted by the cavity is

$$E_t = \frac{t_1 t_2}{1 + r_1 r_2 \exp(-2ikl)} E_i(t) .$$

If we are near resonance we have

$$2kl = (2n+1)\pi + \frac{2l}{c}\Omega ,$$

where Ω is a small variation of the light frequency. Under this approximation, the transmitted field becomes

$$E_t \simeq \frac{t_1 t_2}{1 + r_1 r_2(-1 + i2l\Omega/c)} E_i(t) .$$

Rearranging the previous expression and after some definition we will have the cavity transfer function

$$H(s) = \frac{E_t}{E_i} \simeq T\frac{\gamma}{\gamma + s} \qquad , s = i\Omega + \sigma , \tag{1}$$

where

$$T = \frac{t_1 t_2}{1 - r_1 r_2}, \qquad \gamma = \frac{c}{2l}\frac{1 - r_1 r_2}{r_1 r_2} . \tag{2}$$

The coefficient T is indeed the cavity transmission coefficient when the light is resonant.

As expected, the cavity response corresponds to a first order low pass filter and therefore can be represented in the time domain by the ordinary differential equation

$$E_t(t) + \tau \frac{dE_t(t)}{dt} = T\, E_i \,, \qquad \tau = \frac{1}{\gamma}\,.$$

3 Fabry-Perot Response to an Exponentially Decaying Light

Let's suppose that the input field is

$$E_i(t) = \begin{cases} E_0 & e^{-i\omega t} & t < 0 \\ \\ E_0 & e^{-(\gamma_i + i\omega)t} & t > 0 \end{cases} ,$$

then its Laplace transform will be

$$\begin{aligned} E_i(s) &= E_0 \int_0^{\infty} e^{i\omega t' + t' s} dt' \;+\; E_0 \int_0^{\infty} e^{-(\gamma_i + i\omega)t - ts} dt \\ &= \frac{E_0}{-i\omega\Omega + s} + \frac{E_0}{\gamma_i + i\Omega + s} \\ &= E_0 \frac{\gamma_i + 2s}{(-i\Omega + s)(\gamma_i + i\Omega + s)} \end{aligned}$$

In the Laplace space, the electric field cavity output will be

$$E_t(s) = H(s)\, E(s) = E_0 \frac{\gamma_i + 2s}{(-i\Omega + s)(\gamma_i + i\Omega + s)}\, \beta \frac{\gamma}{\gamma + s}$$

3.1 Transmitted Field

Taking the inverse Laplace transform we finally get for $t > 0$

$$E_t(t) = E_0 T \left\{ \frac{\gamma}{\gamma - \gamma_i + i\Omega} e^{-(\gamma_i + i\omega)t} + \frac{\gamma}{(1 - \gamma/\gamma_i - i\Omega/\gamma_i)(\gamma + i\Omega)}\; e^{-(\gamma + i\omega)t} \right\}$$

If the input light is abruptly interrupted we will have

$$\gamma_i \gg \gamma \qquad \Rightarrow \qquad E_t(t) = E_0 \beta \left\{ \frac{\gamma}{(\gamma + i\Omega)}\; e^{-(\gamma + i\omega)t} \right\}$$

then, the cavity transmitted field cavity is just a pure ring-down.

3.2 Transmitted Power

Supposing that the cavity is perfectly resonant then $\Omega = 0$, and the transmitted field becomes

$$E_t(t) = E_0 T \left(\frac{\gamma}{\gamma - \gamma_i} e^{-\gamma_i t} - \frac{\gamma_i}{\gamma - \gamma_i} \, e^{-\gamma t} \right)$$

Computing the field s intensity we will have

$$I_t(t) = I_0 \frac{T^2}{(\gamma - \gamma_i)^2} \left(\gamma^2 e^{-2\gamma_i t} - 2\gamma\gamma_i \, e^{-(\gamma+\gamma_i)t} + \gamma_i^2 \, e^{-2\gamma t} \right)$$

Rewriting the previous equation in terms of time constants, we will finally have

$$I_t(t) = I_0 \frac{T^2}{(\tau_i - \tau)^2} \left(\tau_i^2 e^{-2t/\tau_i} - 2\tau\tau_i e^{-(1/\tau + 1/\tau_i)t} + \tau^2 e^{-2t/\tau} \right) \tag{3}$$

where

$$\tau = \frac{1}{\gamma} \qquad \tau_i = \frac{1}{\gamma_i} .$$

It is worthwhile to notice that

$$I(t; \tau, \tau_i) = I(t, \tau_i, \tau) .$$

3.3 Time Constant Measurement

Supposing the the input light is abruptly interrupted then we will find once more the expected result

$$\tau_i \ll \tau \qquad \Rightarrow \qquad I_t(t) = I_0 T^2 \, e^{-2t/\tau} .$$

Conversely if

$$\tau \ll \tau_i \qquad \Rightarrow \qquad I_t(t) = I_0 T^2 \, e^{-2t/\tau_i} .$$

In other words, for these simple case, the direct measurable time constant τ_m of the transmitted power decay is half the time constant response of the cavity $\tau_m = \tau/2$ or half of the time constant of the input light $\tau_m = \tau_i/2$.

In the more general case, performing two independent measurements, one of the decay of the input light and another of the decay of the transmitted light will allow to determine the time constant of the cavity.

Another apparent option is to determine both time constants from a single decay time. Anyway, this method is limited by the strong correlation of τ, and τ_i.

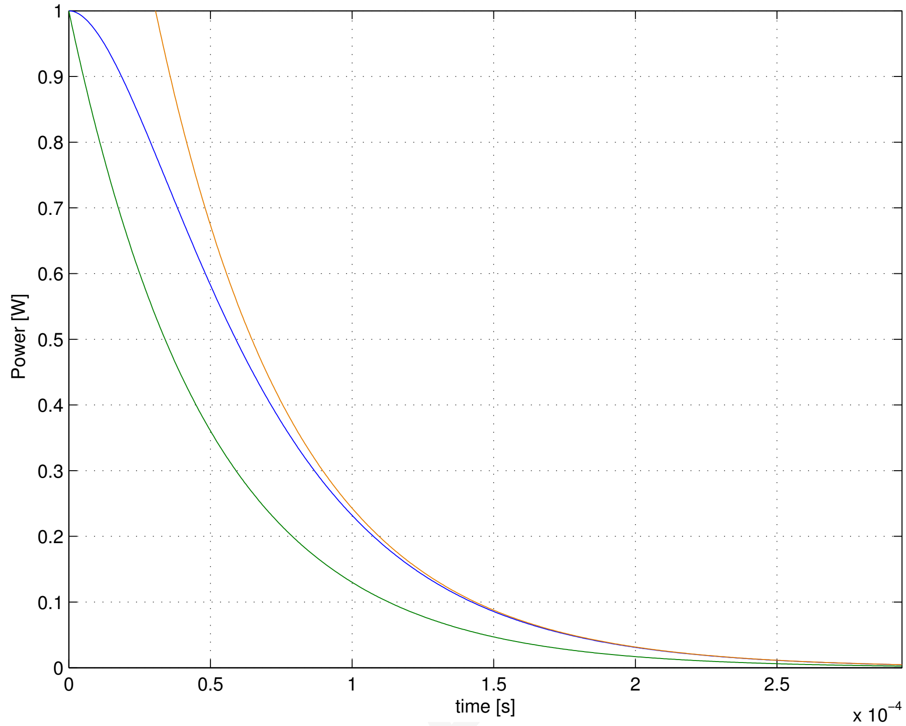


Figure 1: Calculated decays of the cavity transmitted power. Blue curve is the expected decay due to the exponentially decaying of the input light. green and gold curves are with light instantaneously blocked but different initial time.

3.4 Example

Lets consider that case of the Adv LIGO 40meter prototype where the suspended mode cleaner output feeds one of the arms Fabry-Perot. In this case, the expected time constant are

$$\{\tau_{mc}\}_{nominal} = 39.79\mu\text{s} \qquad \{\tau_{FP}\}_{nominal} = 98.04\mu\text{s}$$

Figure 1 clearly shows that the effect of the exponentially decaying input is not negligible at times close to the initial decay (blue and green curves). However, the tail of such more complex decay shows a quite good agreement with a simple exponentially decay if properly time translated as shown by the gold curve.

4 Finesse Versus Time Constant

Considering the classical definition of finesse and eq. 2 we have

$$\gamma = \frac{c}{2l}\frac{1 - r_1 r_2}{r_1 r_2} = \frac{c}{2l}\frac{\pi}{\mathcal{F}} \Rightarrow \qquad \mathcal{F} = \frac{\pi}{2}\frac{c}{l}\frac{1}{\gamma} = \frac{\pi}{2}\frac{c}{l}\tau\,.$$

4.1 Finesse Measurement Resolution Consideration

The relative contribution to the uncertainty on the finesse is

$$\frac{\Delta\mathcal{F}}{\mathcal{F}} = \frac{\Delta l}{l} + \frac{\Delta\tau}{\tau},$$

which means that we have to know the cavity length and the time constant with the same relative uncertainty. Practically, this measurement is limited by the knowledge of the length of the cavity.

For example, assuming an uncertainty of $\Delta l = 1$cm on the 40 m cavity length $l = 38.5$m then τ must be measured with a relative uncertainty of the order $\Delta\tau/\tau \simeq 3\%$. For a time constant or the order of 10μmthe instrumental resolution $\Delta\tau$ should be better than 0.1μs .

5 Measurement of the 40 Fabry-Perot Cavities

Measurement of the power decay was done using a Thorlabs PDA255 photodetector with a declared band-with of 5MHz, connected to a Tektronix ?? 300MHz digital oscilloscope. The photodetector was placed on the transmitted beam of the Y arm.

To block the light we flipped the phase of the PSL FSS frequency servo. Surprisingly, only the mode cleaner looses lock and the PMC stays locked due to the large dynamic range of the PMC PZT and the fast enough PMC servo.

Unfortunately, the PMC servo is too slow to allow measure the time constant of the suspended mode cleaner.

The results obtained fitting 5 times the tail of the are here resumed

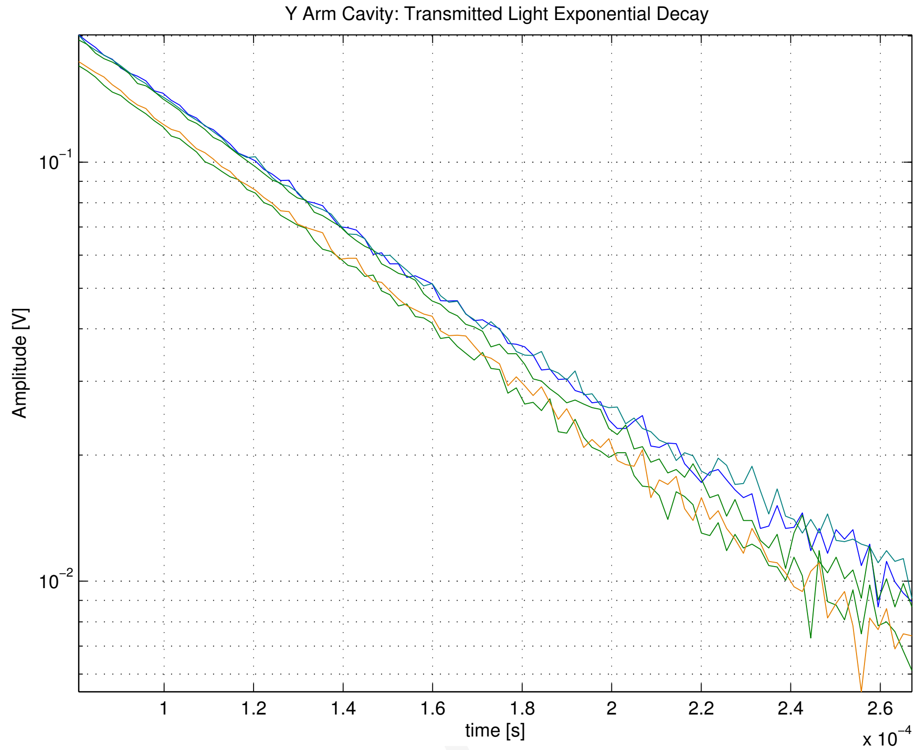
$$\begin{aligned}\tau &= (109.794 \pm 0.322)\mu\text{s} \\ \mathcal{F} &= (1.344 \pm 0.004) \cdot 10^3\end{aligned}$$

All the fits shows a remarkably good agreement with the theoretical exponential decay (worst reduced chi square $\chi^2_{n-p} = 1.6$) and the residuals plots don´t reveal any clear systematic error (see appendix section A.1).

Large enough fluctuations of the arm transmitted power when locked makes very difficult to measure the decay right after the light is blocked.

Considering a past measurement of the finesse of the MC[2] $\mathcal{F}_i = 970$ and fitting most of the exponential decay we obtained

$$\begin{aligned}\tau &= (115.359 \pm 1.412)\mu\text{s} \\ \mathcal{F} &= (1.412 \pm 0.013) \cdot 10^3\end{aligned}$$



The two measures differs by $\Delta\tau = 5.5\mu$s and cannot be accounted by the uncertainties ($\sigma_{\Delta\tau} = 1.44\mu$s). This discrepancy between the two measurements can be explained by the systematic error on the fit (see appendix section A.2) which can come from the lack of the knowledge of the exact ring-down start time (the used fit procedure does not converge if the initial time is not imposed). Moreover, a contribution to this systematic error can come from a systematic error on the measured time constant of the MC.

Even with such a systematic error the fit seems to describe well a large chunk of the decay, where a simple decay model fails.

A Fit Measurements

This appendix shows in details the results of all the fits. Due to the limitations of the fitting procedure some of the parameters had to be kept constant biasing the fit.

A.1 Fit Simple Exponential Decay

This appendix reports the 5 measurements done considering just the exponential decay tail. In this case none of the parameters were kept constant.

A.2 Fit Complex Exponential Decay

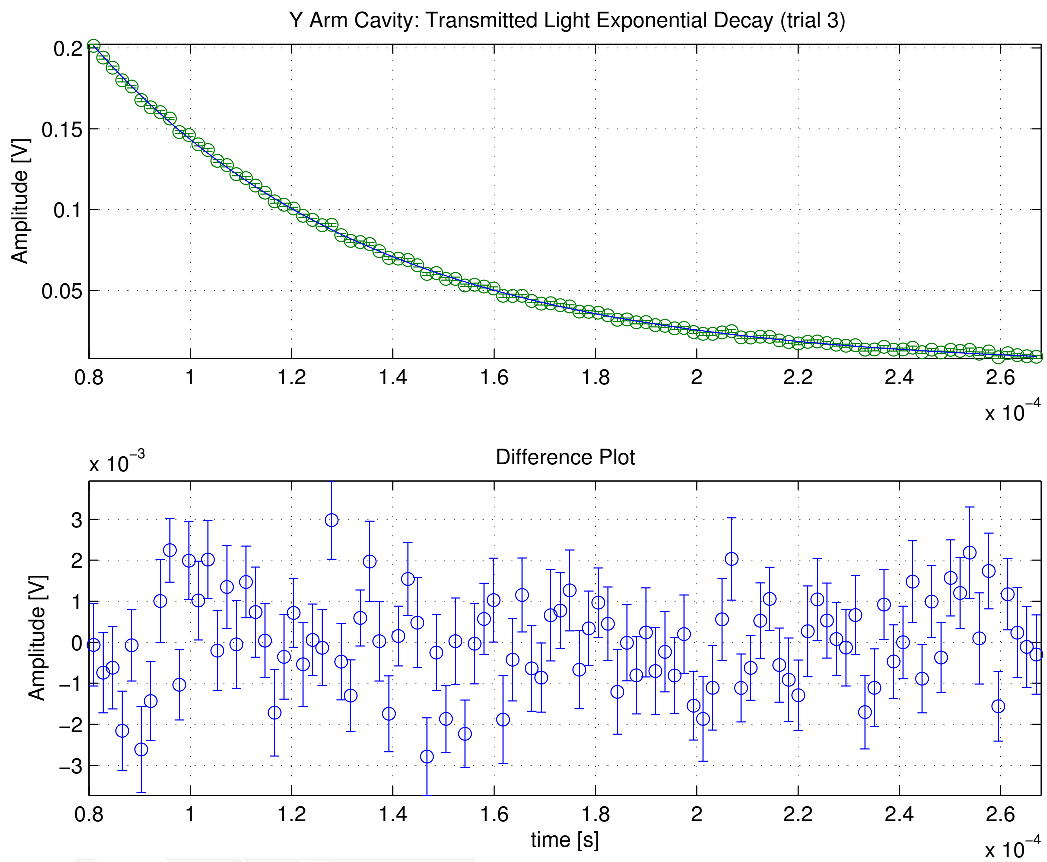
This appendix reports the 5 measurements done considering most of the exponential decay.

Unfortunately, for this complex fitting decay the input field time constant $tau1 = \tau_i$ was kept fixed because of the inability of the fitting procedure to converge. For the same reason some other parameters $t0$ and $I0$ were kept constant. This limitation is probably due to the strong correlation among the parameters.

References

[1] D. Anderson, Joseph C. Frisch , and Carl S. Masser, *Mirror reflectometer based on cavity decay time*, Applied Optics 23-8 1984.

[2] Osamu Miyakawa , private communication.



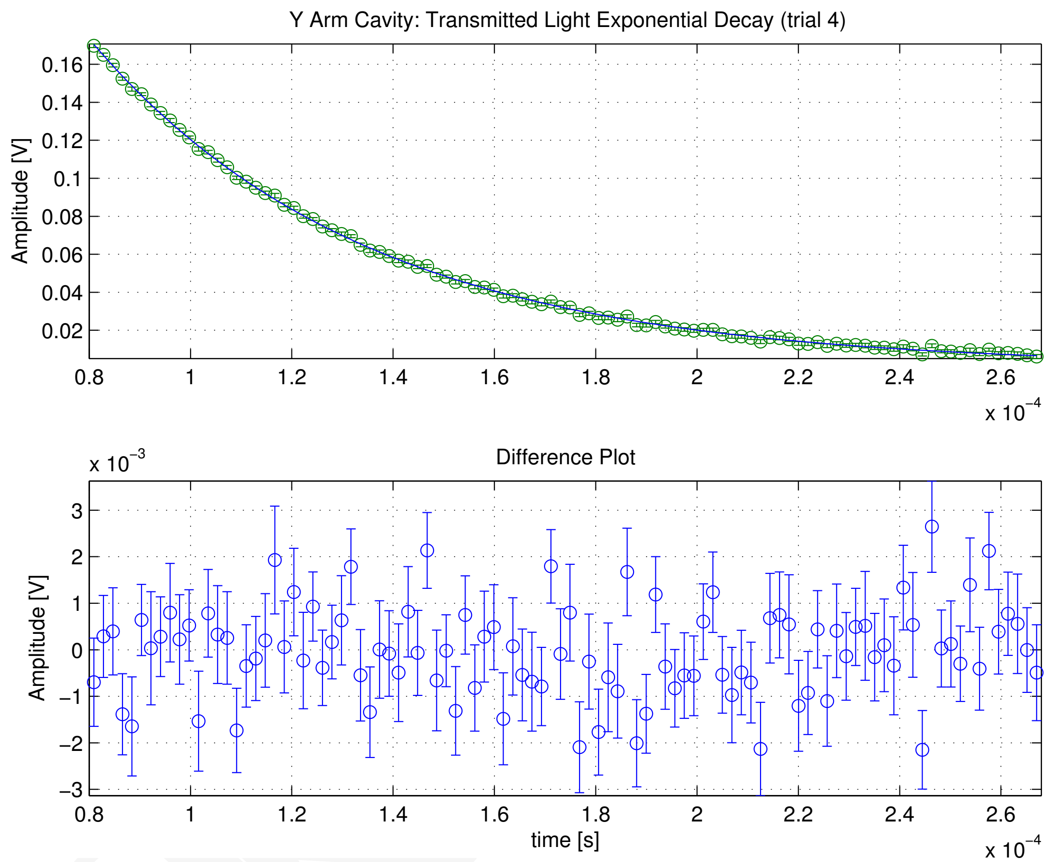

```
Fitting Function:
 y(x)=I0*exp(-2*(x-t0)/tau)+c

Parameters:
                      tau =   1.10831e-04 +-   7.73600e-07
                       I0 =   6.50000e-01 +-   1.72700e+09
                        c =   2.31185e-03 +-   3.71200e-04
                       t0 =   1.53582e-05 +-   1.47200e+05

Degrees of Freedom = 96

Reduced ChiSquare = 1.62884

Correlation Matrix:
                      tau  1.00
                       I0  0.32  1.00
                        c -0.82  0.02  1.00
                       t0 -0.32 -1.00 -0.02  1.00
```



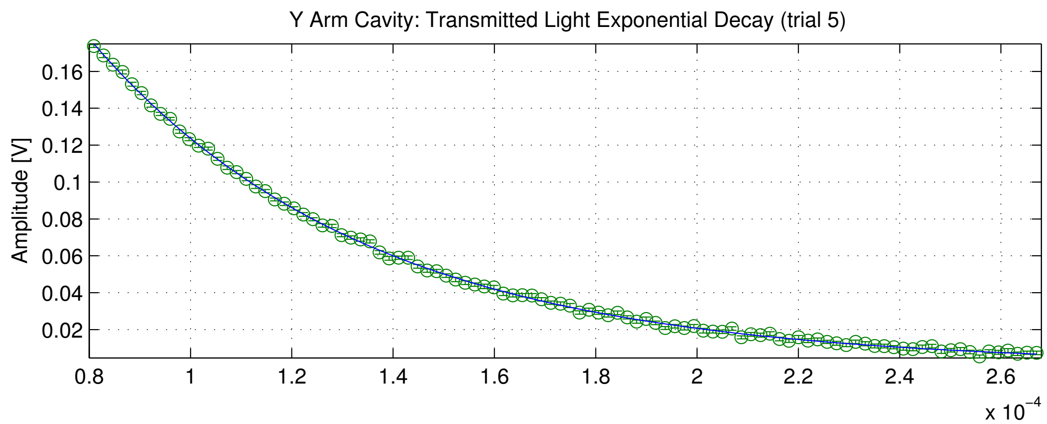

```
Fitting Function:
 y(x)=I0*exp(-2*(x-t0)/tau)+c

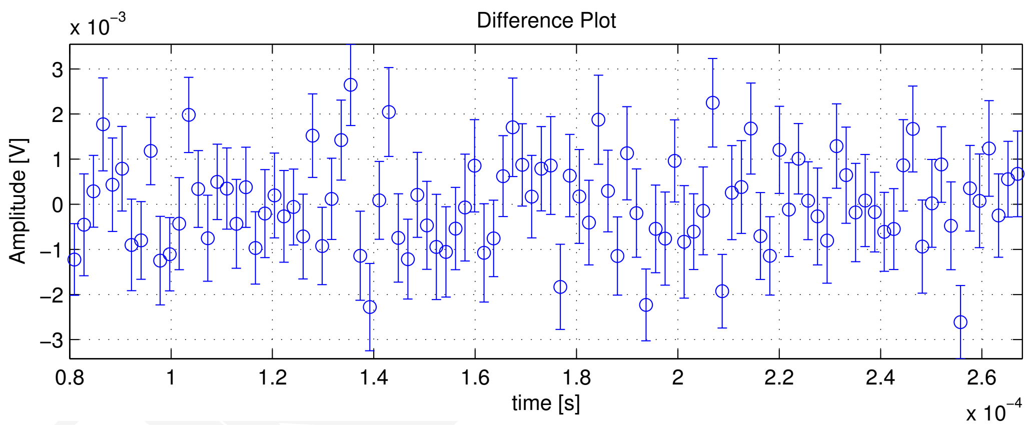
Parameters:
                      tau =   1.08867e-04 +-   7.29800e-07
                       I0 =   6.50000e-01 +-   6.18700e+08
                        c =   1.07411e-03 +-   3.11800e-04
                       t0 =   7.72605e-06 +-   5.18100e+04

Degrees of Freedom = 96

Reduced ChiSquare = 1.20808

Correlation Matrix:
                      tau  1.00
                       I0  0.23  1.00
                        c -0.87 -0.10  1.00
                       t0 -0.23 -1.00  0.10  1.00
```





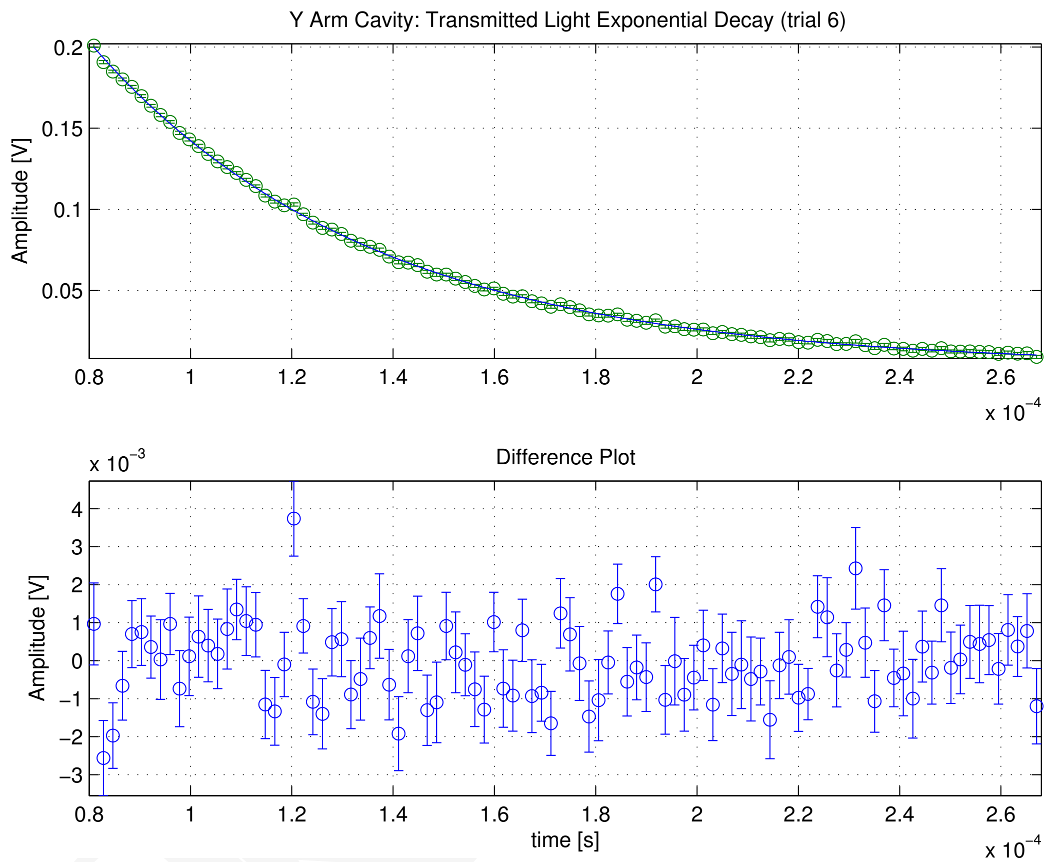

```
Fitting Function:
 y(x)=I0*exp(-2*(x-t0)/tau)+c

Parameters:
                      tau =   1.09460e-04 +-   8.17200e-07
                       I0 =   6.50000e-01 +-   1.04300e+09
                        c =   9.26683e-04 +-   3.30300e-04
                       t0 =   8.79009e-06 +-   8.77700e+04

Degrees of Freedom = 96

Reduced ChiSquare = 1.32072

Correlation Matrix:
                      tau  1.00
                       I0 -0.44  1.00
                        c -0.81  0.07  1.00
                       t0  0.44 -1.00 -0.07  1.00
```



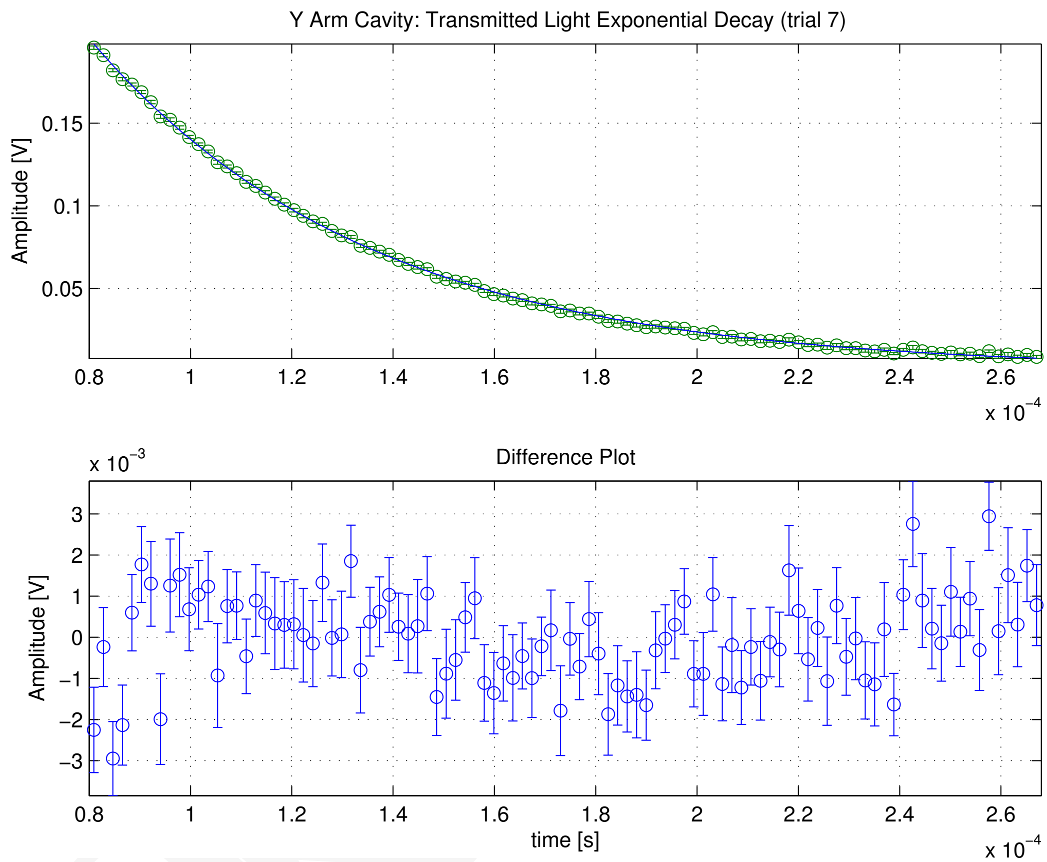

```
Fitting Function:
 y(x)=I0*exp(-2*(x-t0)/tau)+c

Parameters:
                      tau =   1.09899e-04 +-   6.48300e-07
                       I0 =   6.50000e-01 +-   1.59300e+09
                        c =   3.67065e-03 +-   3.21500e-04
                       t0 =   1.51031e-05 +-   1.34700e+05

Degrees of Freedom = 96

Reduced ChiSquare = 1.28869

Correlation Matrix:
                      tau  1.00
                       I0 -0.13  1.00
                        c -0.87 -0.02  1.00
                       t0  0.13 -1.00  0.02  1.00
```



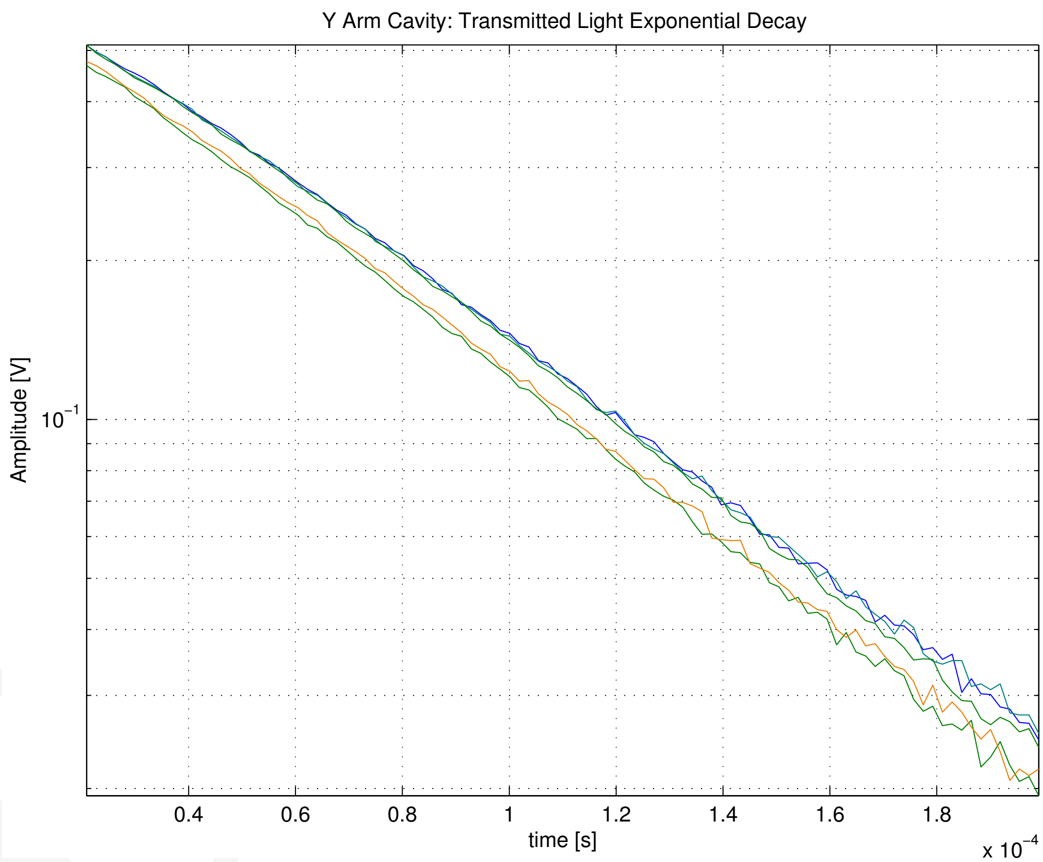

```
Fitting Function:
 y(x)=I0*exp(-2*(x-t0)/tau)+c

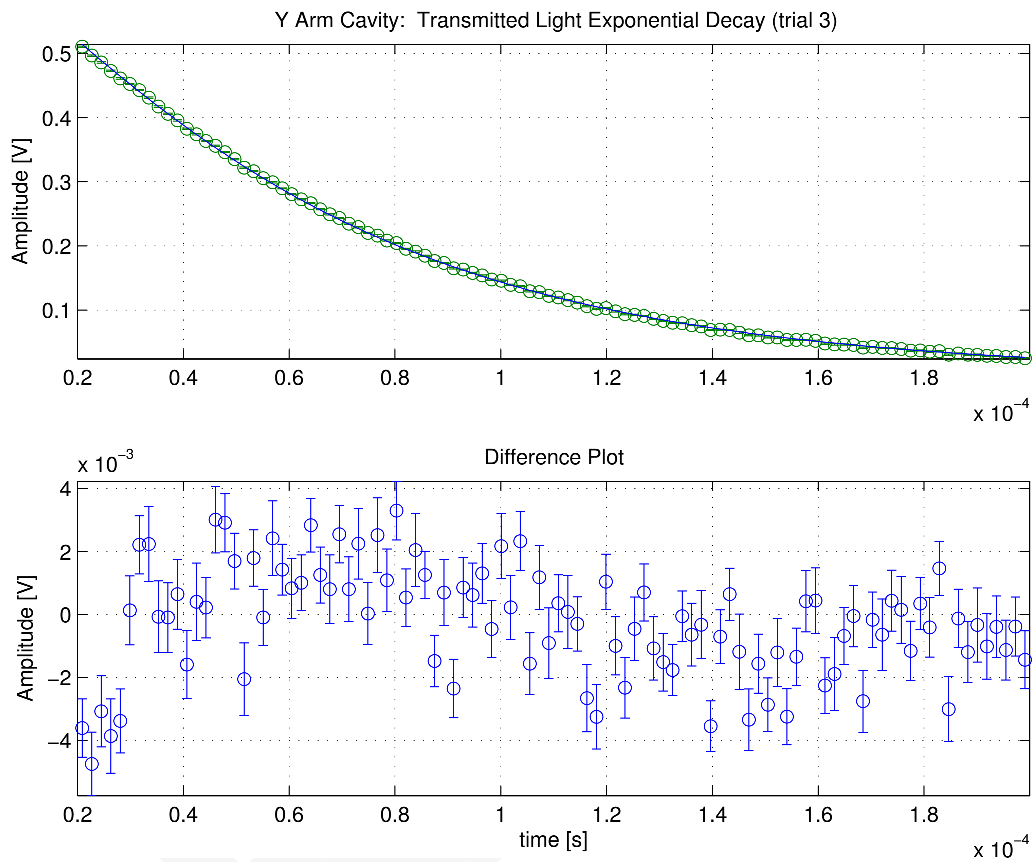
Parameters:
                      tau =  1.09914e-04 +-  7.32900e-07
                       I0 =  6.50000e-01 +-  1.55000e+09
                        c =  1.24741e-03 +-  3.49000e-04
                       t0 =  1.52208e-05 +-  1.31000e+05

Degrees of Freedom = 96

Reduced ChiSquare = 1.37253

Correlation Matrix:
                      tau  1.00
                       I0 -0.35  1.00
                        c -0.86  0.12  1.00
                       t0  0.35 -1.00 -0.12  1.00
```


Y Arm Cavity: Transmitted Light Exponential Decay
Amplitude [V]
10^-1
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
time [s]
x 10^-4



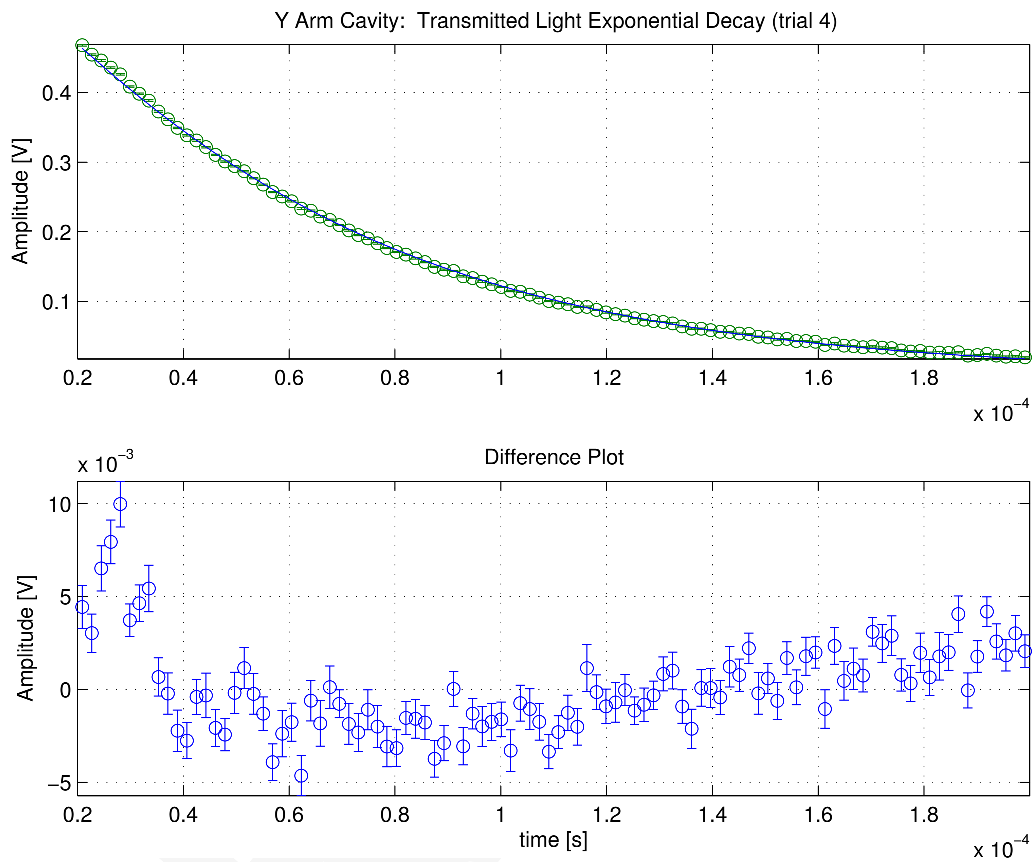

```
Fitting Function:
 y(x)=I0*(tau1^2*exp(-2*(x-t0)/tau1)-2*tau*tau1*exp(-(1/tau+1/tau1)*(x-t0))+tau^2*exp(-2*(x-t0)/tau))+I1

Parameters:
                      tau =   1.13479e-04 +-   2.05400e-07
                     tau1 =   2.57300e-05 (Constant)
                       I0 =   9.78864e+07 +-   6.12100e+05
                       I1 =   1.00000e-03 (Constant)
                       t0 =  -2.30000e-05 (Constant)

Degrees of Freedom = 98

Reduced ChiSquare = 3.40579

Correlation Matrix:
                      tau   1.00
                     tau1   0.00   1.00
                       I0  -0.99   0.00   1.00
                       I1   0.00   0.00   0.00   1.00
                       t0   0.00   0.00   0.00   0.00   1.00
```



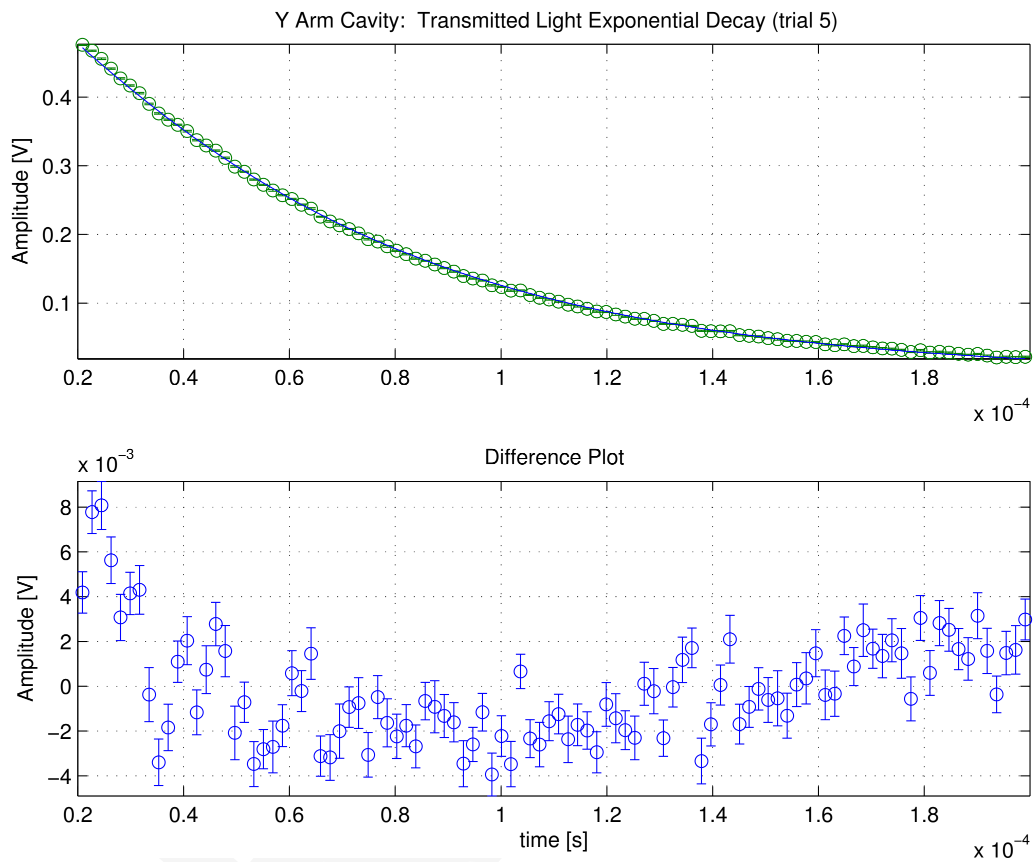

```
Fitting Function:
 y(x)=I0*(tau1^2*exp(-2*(x-t0)/tau1)-2*tau*tau1*exp(-(1/tau+1/tau1)*(x-t0))+tau^2*exp(-2*(x-t0)/tau))+I1

Parameters:
                      tau =   1.13226e-04 +-   3.00900e-07
                     tau1 =   2.57300e-05 (Constant)
                       I0 =   9.78864e+07 +-   9.23900e+05
                       I1 =  -5.00000e-03 (Constant)
                       t0 =  -2.90000e-05 (Constant)

Degrees of Freedom = 98

Reduced ChiSquare = 5.90984

Correlation Matrix:
                      tau   1.00
                     tau1   0.00   1.00
                       I0  -0.99   0.00   1.00
                       I1   0.00   0.00   0.00   1.00
                       t0   0.00   0.00   0.00   0.00   1.00
```




```
Fitting Function:
 y(x)=I0*(tau1^2*exp(-2*(x-t0)/tau1)-2*tau*tau1*exp(-(1/tau+1/tau1)*(x-t0))+tau^2*exp(-2*(x-t0)/tau))+I1

Parameters:
                      tau =  1.13290e-04 +-  2.89200e-07
                     tau1 =  2.57300e-05 (Constant)
                       I0 =  9.78864e+07 +-  8.79000e+05
                       I1 = -4.00000e-03 (Constant)
                       t0 = -2.80000e-05 (Constant)

Degrees of Freedom = 98

Reduced ChiSquare = 5.97972

Correlation Matrix:
                      tau  1.00
                     tau1  0.00  1.00
                       I0 -0.99  0.00  1.00
                       I1  0.00  0.00  0.00  1.00
                       t0  0.00  0.00  0.00  0.00  1.00
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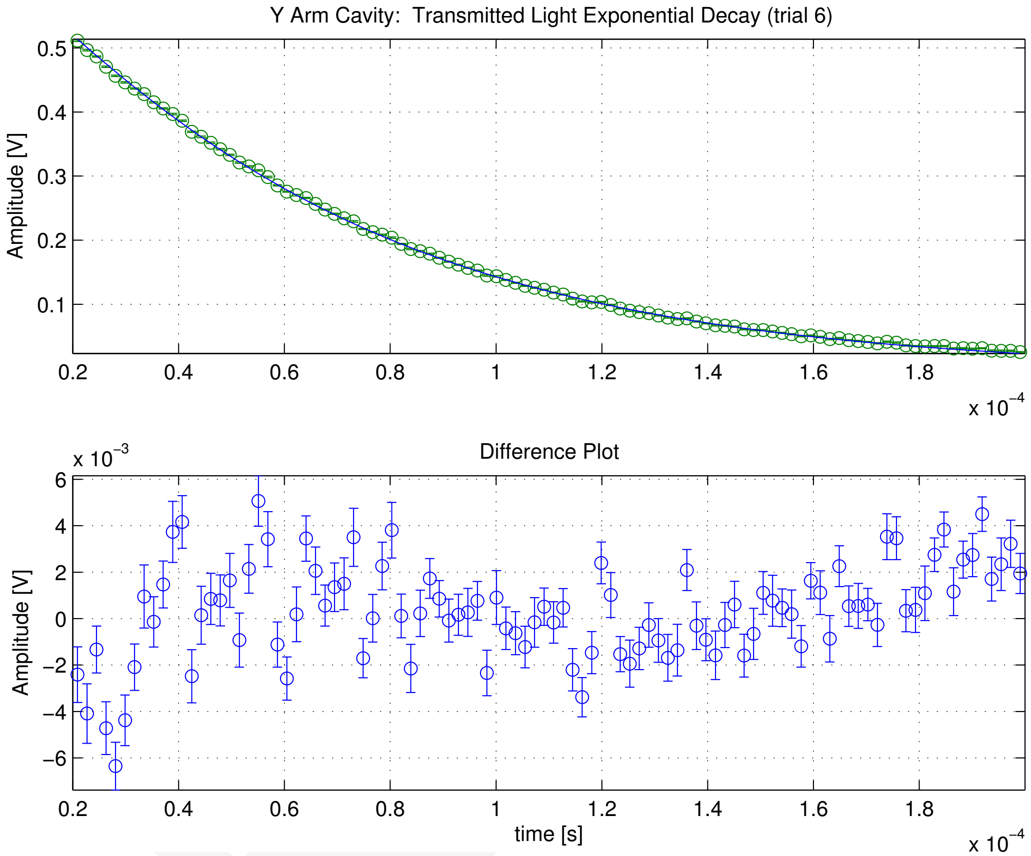


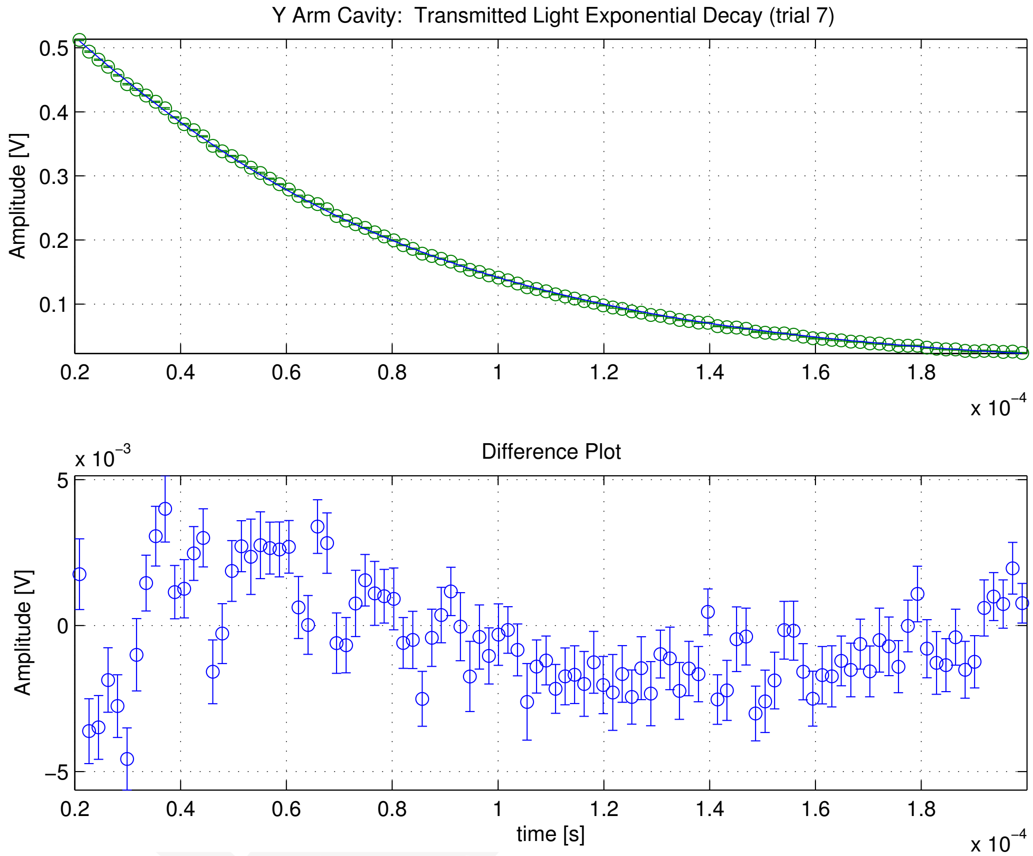

```
Fitting Function:
 y(x)=I0*(tau1^2*exp(-2*(x-t0)/tau1)-2*tau*tau1*exp(-(1/tau+1/tau1)*(x-t0))+tau^2*exp(-2*(x-t0)/tau))+I1

Parameters:
                      tau =  1.17581e-04 +-  2.46900e-07
                     tau1 =  2.57300e-05 (Constant)
                       I0 =  9.78864e+07 +-  7.29700e+05
                       I1 = -4.00000e-03 (Constant)
                       t0 = -3.00000e-05 (Constant)

Degrees of Freedom = 98

Reduced ChiSquare = 4.71878

Correlation Matrix:
                      tau  1.00
                     tau1  0.00  1.00
                       I0 -0.99  0.00  1.00
                       I1  0.00  0.00  0.00  1.00
                       t0  0.00  0.00  0.00  0.00  1.00
```




```
Fitting Function:
 y(x)=I0*(tau1^2*exp(-2*(x-t0)/tau1)-2*tau*tau1*exp(-(1/tau+1/tau1)*(x-t0))+tau^2*exp(-2*(x-t0)/tau))+I1

Parameters:
                      tau =   1.17339e-04 +-   2.13900e-07
                     tau1 =   2.57300e-05 (Constant)
                       I0 =   9.78864e+07 +-   6.28500e+05
                       I1 =  -4.00000e-03 (Constant)
                       t0 =  -3.00000e-05 (Constant)

Degrees of Freedom = 98

Reduced ChiSquare = 3.37659

Correlation Matrix:
                      tau  1.00
                     tau1  0.00  1.00
                       I0 -0.99  0.00  1.00
                       I1  0.00  0.00  0.00  1.00
                       t0  0.00  0.00  0.00  0.00  1.00
```