

## Rate per unit volume for uniform distribution

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The plots we have been presenting address the question: “what is the upper limit on the number of coincident events as a function of peak amplitude of the burst, averaged over source direction and polarization”. We can think of this as an average over sources of fixed luminosity, distributed over the surface of a sphere of fixed radius.

The resultant efficiency is shown as the black curve below, for triple coincidence.

$$\langle \varepsilon_{ab\dots} \rangle (h) \equiv \bar{\varepsilon}(h) \equiv \int d \cos \theta d\phi d\psi \varepsilon_a(R_a(\theta, \phi, \psi)h) \varepsilon_b(R_b(\theta, \phi, \psi)h) \dots \quad (1)$$

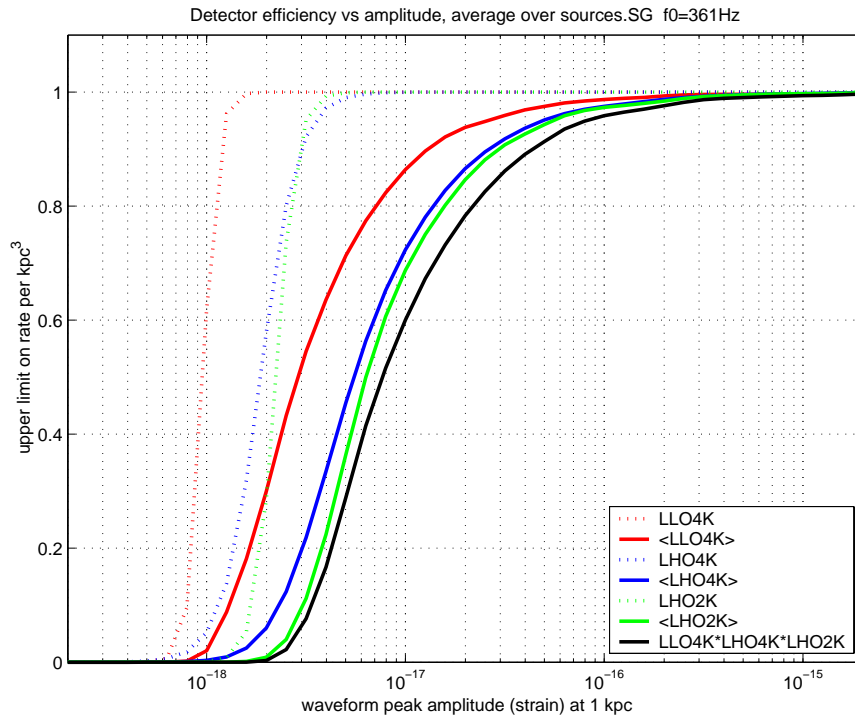


Figure 1: Efficiency  $\bar{\varepsilon}(h)$  for sine-gaussians with  $f_0 = 361$  Hz, for S1 triple coincidence, averaged over source direction and polarization.

But we can ask a different (better?) question: “what is the upper limit on the rate of bursts per unit volume, as a function of intrinsic luminosity,

assuming uniform density of sources in space?” That is, an average over sources of fixed luminosity, distributed uniformly over the volume of space. Many people have asked for the results to be expressed in this way.

We assume a fixed intrinsic luminosity that gives a peak strain amplitude of  $h$  at a distance of  $r_0 = 1$  kpc (chosen arbitrarily). The rate per unit volume, which we want to determine, is  $R(h)$ . We think of this as a function of intrinsic luminosity  $hr_0$  (strain at 1 kpc). At some arbitrary distance  $r$ , the peak amplitude is  $hr_0/r$  and the triple-coincidence detection efficiency for that amplitude, averaged over all source directions and polarizations, is  $\bar{\varepsilon}(hr_0/r)$ .

Assume that the rate per unit mass is  $\mathcal{R}$ , and the mass per unit volume is  $\rho(\vec{r})$ . The total detected rate per unit volume  $\dot{N} = N/livetime$ , which we measure over an observation time  $t = livetime$ , is then:

$$\dot{N} = \int_0^\infty d\vec{r} \mathcal{R} \rho(\vec{r}) \varepsilon_a(R_a(\theta, \phi, \psi) hr_0/r) \varepsilon_b(R_b(\theta, \phi, \psi) hr_0/r) \cdots \quad (2)$$

This integral can be done via a simple Monte Carlo, just as has been done for the case of fixed amplitude, assuming some model of the mass density  $\rho(\vec{r})$  in our galaxy (and beyond).

When  $\vec{r}$  is on the scale of the dimensions of the galaxy,  $\rho(\vec{r})$  becomes a strong function of  $\vec{r}$ . But for the sake of a simple analytical solution, we can approximate it as  $\rho(\vec{r}) = \rho_0$ . We'd then like to obtain a measurement or upper limit on the rate per unit volume,  $\mathcal{R}\rho_0$ . With the assumption of uniform mass density,  $\rho(\vec{r}) = \rho_0$ ,

$$\dot{N} = \mathcal{R}\rho_0 \int_0^\infty d\vec{r} \varepsilon_a(R_a(\theta, \phi, \psi) hr_0/r) \varepsilon_b(R_b(\theta, \phi, \psi) hr_0/r) \cdots \quad (3)$$

The integral over angle, at fixed  $hr_0/r$ , has already been done, via a simple Monte Carlo, using eqn. 1. Thus we have:

$$\dot{N} = 4\pi \mathcal{R}\rho_0 \int_0^\infty r^2 dr \bar{\varepsilon}(hr_0/r) \quad (4)$$

To evaluate the integral, set  $v = \frac{hr_0}{hr_0}$ , where  $h_{max}$  is the strain above which our detection efficiency is 1:  $\bar{\varepsilon}(h > h_{max}) = 1$ . Then,

$$\dot{N} = 4\pi \mathcal{R}\rho_0 \left( \frac{hr_0}{h_{max}} \right)^3 \int_0^\infty v^2 dv \bar{\varepsilon}(h_{max}/v) \quad (5)$$

$$\equiv 4\pi \mathcal{R}\rho_0 \left( \frac{hr_0}{h_{max}} \right)^3 I \quad (6)$$

The integral  $I$  is independent of the assumed intrinsic luminosity  $hr_0$ .  $I$  is the only thing in this expression that depends on the detector. It can easily be evaluated numerically, given our known (measured through LDAS simulation) and parameterized  $\bar{\varepsilon}(h_{max}/v)$ . That integral defines an effective luminosity reach  $h_{eff}r_0$ ; so all our knowledge of our detector performance boils down to this one quantity.

$$\frac{1}{3} \left( \frac{h_{max}}{h_{eff}} \right)^3 \equiv \int_0^\infty v^2 dv \bar{\varepsilon}(h_{max}/v) \quad (7)$$

$$\dot{N} = \frac{4\pi r_0^3}{3} \mathcal{R}\rho_0 \left( \frac{h}{h_{eff}} \right)^3 \quad (8)$$

And then, our rate per unit volume is:

$$\mathcal{R}\rho_0 = \frac{3\dot{N}}{4\pi r_0^3} \left( \frac{h_{eff}}{h} \right)^3 \quad (9)$$

We obtain a simple result: a measurement of (or limit on) the burst rate per unit volume,  $\mathcal{R}\rho_0$ , in bursts/second/kpc<sup>3</sup>. We have assumed uniform mass density, and bursts with a single fixed luminosity. We arbitrarily choose  $r_0 = 1$  kpc, so that the luminosity is given by a strain  $h$  at 1 kpc. Under these assumptions, we predict a distribution of observed strain amplitudes  $h$  that goes like  $h^{-3}$ . (Note that these assumptions are astrophysically unreasonable, and that the expected distribution of strain amplitudes will therefore depart from this prediction).

We observe a rate of  $\dot{N}$  events/sec during our observation time. The only quantity that must be determined from the detector network performance is  $h_{eff}$ . Thus, under these assumptions, the search result is characterized by 2 numbers:  $\dot{N}$  and  $h_{eff}$ .

To determine  $h_{eff}$ , we evaluate the integral:

$$I \equiv \frac{1}{3} \left( \frac{h_{max}}{h_{eff}} \right)^3 \equiv \int_0^\infty v^2 dv \bar{\varepsilon}(h_{max}/v) \quad (10)$$

$$= \int_0^1 v^2 dv + \int_1^{h_{max}/h_{min}} v^2 dv \bar{\varepsilon}(h_{max}/v) \quad (11)$$

$$= \frac{1}{3} + \int_1^{h_{max}/h_{min}} v^2 dv \bar{\varepsilon}(h_{max}/v) \quad (12)$$

We know that  $\bar{\varepsilon}$  is a smooth function of the logarithm of the strain, reaching 1 for  $h > h_{max}$  and 0 for  $h < h_{min}$ . It is thus convenient to evaluate this integral after converting to a logarithmic scale. Defining  $x = \log_{10} v$ ,

$$\frac{1}{3} \left( \frac{h_{max}}{h_{eff}} \right)^3 = \frac{1}{3} + \int_0^{\log_{10}(h_{max}/h_{min})} 10^{3x} dx \bar{\varepsilon}(10^{-x} h_{max}) \quad (13)$$

This integral can be done numerically, once for each waveform, yielding, for S1, the table below:

waveform	$h_{eff}$
GA $\tau = 1.0$ msec	$2.0 \times 10^{-17}$
GA $\tau = 2.5$ msec	$9.7 \times 10^{-17}$
SG $f_0 = 153$ Hz	$1.2 \times 10^{-17}$
SG $f_0 = 235$ Hz	$4.8 \times 10^{-18}$
SG $f_0 = 361$ Hz	$4.6 \times 10^{-18}$
SG $f_0 = 554$ Hz	$5.4 \times 10^{-18}$
SG $f_0 = 850$ Hz	$7.5 \times 10^{-18}$
SG $f_0 = 1304$ Hz	$1.5 \times 10^{-17}$
SG $f_0 = 2000$ Hz	$4.4 \times 10^{-17}$

Although the individual IFO efficiency curves are well modeled by sigmoids in  $\log(h)$ , the 3-IFO coincidence efficiency averaged over the sky departs from a sigmoid. Nevertheless, the curves are similar for all 9 waveforms included in the table above. The resulting value of  $h_{eff}$  is always where the 3-IFO coincidence efficiency curve reaches 20% – 25%.