Gravitational wave interferometry: how does it work?

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ABSTRACT

The existence of gravitational waves is predicted by the general relativity theory, but they have not yet been measured directly. The difficulty with these measurements is the very small signals expected from even the strongest sources (astrophysical events such as supernovae). Detection of gravitational waves by interferometric detectors requires resolution of very small displacements, which in turn requires very long arm lengths. In the case of the Laser Interferometer Gravitational-Wave Observatory (LIGO), Michelson interferometers with arm lengths of 4 km and position accuracy on the order of 10^{-18} meters are to be employed. LIGO, funded by the US National Science Foundation, is being constructed at two sites in the United States with initial observation planned in 2002. An overview of the LIGO design requirements, configuration, and control scheme is presented. The optical configuration is discussed in general with particular attention to the characteristics of the core optics. In addition, a brief overview of large-scale gravitational wave observation projects worldwide is presented.

Keywords: Interferometry, gravitational waves, astrophysics, LIGO

1. INTRODUCTION

According to general relativity theory, dense concentrations of energy should strongly warp spacetime. Whenever such an energy concentration changes shape, it should create a corresponding spacetime warpage that propagates out through the Universe at the speed of light. This propagating warpage is called a gravitational wave. Even the strongest astrophysical events will require very small spatial resolution in order to detect the warpage of spacetime on Earth.

Large-scale interferometry is one method of detecting gravitational waves. Ground based interferometers can be effective in the range from 1 to 10^4 Hz, which is the upper end of the frequency band of predicted astrophysical sources[1]. The specific geometric and optical configuration of the interferometer determines the frequency of maximum sensitivity. The LIGO interferometers are operated in a Michelson configuration. The orthogonal geometry of this configuration makes it ideally suited to gravitational wave detection. The gravitational wave strain h will act on each arm of length L independently. The signal is then proportional to the differential length $\sim\Delta L/L$ of the two arms. The position accuracy of LIGO is designed to be on the order of $\Delta L \sim 10^{-18}$ meters rms at roughly 100Hz.

The Laser Interferometer Gravitational-Wave Observatory (LIGO) consists of three Michelson interferometers installed at two different locations within the United States. In order to obtain the sensitivity goals mentioned above, the strategy is:

- Use very long arm lengths, in vacuum, with light storage times of about 1ms.
- Use very low loss materials (both optically and mechanically) as test masses, coupled with high input and resonating power.
- Place the test masses on multi-layer seismic isolation stacks and suspend them in a pendulum configuration.

All of this is then placed under the control of a sophisticated alignment sensing control system, which maintains interferometer lock via a servo network.

Observation is scheduled to begin in 2002; the project is well underway toward realizing that goal. Many of the essential technologies used in LIGO have been demonstrated on interferometers on a smaller scale. One of these is the 40 meter interferometer at Caltech, which has been operating in various optical configurations since 1982[2]. Installation is currently underway for the first two of the three full-scale interferometers. Many of the subsystems have come in with designs and

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hardware that have even better performance than required. This sheds an optimistic light on plans for next generation detectors, which should improve detection sensitivity by more than one order of magnitude.

LIGO is planned as a part of an international network of gravity-wave detectors. Correlation between data from the three LIGO interferometers will be used to eliminate local noise, to detect one of the two gravitational wave waveforms and to locate the source within an annulus on the sky. As a part of a worldwide system of detectors, the network will be able to locate the source (via time delay plus the inteferometers' beam patterns.) within a 2-dimensional error box of roughly one degree[1].

2. ASTROPHYSICAL INTEREST

Although gravitational waves have not yet been detected directly, their influence has been seen and measured with such accuracy that even the Nobel Prize Committee has blessed their reality. In 1993 the Prize was awarded to Hulse and Taylor for their discovery of the binary pulsar PSR 1913+16[3], and for Taylor's observational demonstration that the binary's two neutron stars are spiraling together at the rate predicted by general relativity. The computed and observed inspiral rates agree to within the experimental accuracy, better than one per cent.

2.1 Sources

There are several gravitational wave projects currently underway throughout the world. These are partly driven by the desire to measure gravitational waves directly. More important though, is the use of these waves to probe the Universe and the relativistic nature of gravity.

There is an enormous difference between gravitational waves, and the electromagnetic waves on which our present knowledge of the Universe is based. The following excerpt from Thorne[1] summarizes these differences:

- "Electromagnetic waves are oscillations of the electromagnetic field that propagate through spacetime; gravitational waves are oscillation of the "fabric" of spacetime itself.

- Astronomical electromagnetic waves are almost always incoherent superpositions of emission from individual electrons, atoms, or molecules. Cosmic gravitational waves are produced by coherent, bulk motions of huge amounts of mass-energy--either material mass, or the energy of vibrating, nonlinear spacetime curvature.

- Since the wavelengths of electromagnetic waves are small compared to their sources (gas clouds, stellar atmospheres, accretion disks,…), from the waves we can make pictures of the sources. The wavelengths of cosmic gravitational waves are comparable to or larger than their coherent, bulk-moving sources, so we cannot make pictures from them. Instead, the gravitational waves are like sound; they carry, in two independent waveforms, a stereophonic, symphony-like description of their sources.

- Electromagnetic waves are easily absorbed, scattered, and dispersed by matter. Gravitational waves travel nearly unscathed through all forms and amounts of intervening matter.

- Astronomical electromagnetic waves have frequencies that begin at $f{\sim}10^7$ Hz and extend on upward by roughly 20 orders of magnitude. Astronomical gravitational waves should begin at $\sim 10^4$ Hz (1000-fold lower than the lowest-frequency astronomical electromagnetic waves), and should extend on downward from there by roughly 20 orders of magnitude.

These enormous differences make it likely that:

- The information brought to us by gravitational waves will be very different from (almost "orthogonal to") that carried by electromagnetic waves; gravitational waves will show us details of the bulk motion of dense concentrations of energy, whereas electromagnetic waves show us the thermodynamic state of optically thin concentrations of matter.

- Most (but not all) gravitational-wave sources that our instruments detect will not be seen electromagnetically, and conversely, most objects observed electromagnetically will never be seen gravitationally. Typical electromagnetic sources are stellar atmospheres, accretion disks, and clouds of interstellar gas—none of which emit significant gravitational waves. While typical gravitational wave sources are the cores of supernovae (which are hidden from electromagnetic view by dense layers of surrounding stellar gas), and colliding black holes (which emit no electromagnetic waves at all).

- Gravitational waves may bring us great surprises. In the past, when a radically new window has been opened onto the Universe, the resulting surprises have had a profound, indeed revolutionary, impact. For example, the radio universe, as discovered in the 1940s, 50s and 60s, turned out to be far more violent than the optical Universe; radio waves brought us quasars, pulsars, and the cosmic microwave radiation, and with them our first direct observational evidence for black holes, neutron stars, and the heat of the big bang. It is reasonable to hope that gravitational waves will bring a similar "revolution"."

3. DETECTION LEVELS

In order to radiate strongly a source must have a very large, non-spherical, internal kinetic energy. The best known way to achieve a huge internal kinetic energy is via gravity. According to energy conservation, any gravitationally induced kinetic energy must be of the order of the source's gravitational potential energy. A huge potential energy, in turn requires that the source be compact, not much larger than its own gravitational radius. Thus, the strongest gravity-wave sources must be highly compact, dynamical concentrations of large amounts of mass, for example colliding and coalescing black holes and neutron stars. The strength of the wave *h* from a gravitational wave source can be approximated by[1]

$$h \sim \frac{1}{c^2}\frac{4G(E_{kin}^{ns}/c^2)}{r} \tag{1}$$

Where E_{kin}^{ns} is the internal non-spherical kinetic energy of the source, r is the distance of the source from Earth and G and c are Newton's gravitation constant and the speed of light. Thus, h is about 4 times the gravitational potential produced at Earth by the mass equivalent of the sources non-spherical, internal kinetic energy, made dimensionless by dividing by c^2.

Such sources cannot remain highly dynamic for long; their motions will be stopped by energy loss to gravitational waves and/or the formation of a black hole. The strongest sources should therefore be transient. They should also be very rare, so rare that to see a reasonable event rate will require the sensitivity to detect outward through a substantial fraction of the Universe.

For highly compact, dynamical objects that radiate in the high-frequency band, the internal non-spherical kinetic energy E_{kin}^{ns}/c^2 is of the order of the mass of the Sun. This gives $h \sim 10^{-22}$ for such sources at the Hubble distance (3000 Mpc), h^{-20} at the Virgo cluster of galaxies (15 Mpc) and $h \sim 10^{-17}$ in the outer reaches of our own Milky Way galaxy (20 kpc). These numbers set the scale of sensitivities required of ground based interferometers. The effect of the gravitational wave on earth is seen by an interferometer as $h \sim \Delta L/L$. Thus to achieve wave sensitivities of 10^{-21} to 10^{-22} the interferometer measurement accuracy must be of order $\Delta L \sim 10^{-18}$ meters assuming arm lengths of 1 to 10km.

4. HOW DOES IT WORK?

The two LIGO observatory sites are located at nearly opposite ends of the United States, in Hanford Washington and Livingston Louisiana. The site locations and relative arm orientations are shown in Figure 1. The observatory at Hanford is comprised of two interferometers, a 2 km long system and a 4 km long system. Each interferometer has unique lasers and optics but the two will share the same vacuum enclosure. The observatory at Livingston will house a single 4 km long system. There is room for growth as each of the facilities is designed to accommodate several additional interferometers occupying the same vacuum system.

Figure 1: The relative positions of the two LIGO observatories and the orientation of their arms.

4.1 Interferometry

Each LIGO interferometer is fundamentally a suspended Michelson interferometer. The interferometers are operated in vacuum. Figure 2 shows the initial LIGO optical layout. Fabry-Perot cavities are formed within each of the Michelson arms. The mirrors that form the Fabry-Perot cavities are called test masses; these will be discussed later in detail. The lengths of the Fabry-Perot cavities change when a gravitational wave is incident on the detector. This shifts the resonant frequency of the cavities relative to the laser frequency, changing the phase of the light in the cavity as well as the phase of the light that exits from the cavity toward the beam splitter. The relative phase of the two beams recombined at the beamsplitter changes by an amount proportional to the differential change in cavity length. This phase shift causes a change in the intensity of the recombined light, it is this signal that is monitored, and is directly proportional to the gravitational-wave strain $h(t)$.

Any light that is not absorbed or scattered in the arms is returned toward the laser. A recycling mirror is placed in the input path to return this would-be wasted light back to the interferometer. The position and reflectivity of the mirror are chosen such that incoming light adds constructively with light circulating in the interferometer, but light reflected by the mirror back toward the laser is combined destructively with the light transmitted by this mirror from the interferometer. This is the same as using a more powerful laser source as it can result in an additional gain of about 30 in power.

The interferometer is illuminated by a 10 Watt Nd:YAG laser. The laser is composed of a master oscillator and two pass amplifier. The laser operates at a single frequency, and has both frequency and intensity stabilization loops. The frequency is locked to a fixed reference cavity. This reference cavity is temperature stabilized, and is a part of the servo loop that changes the frequency of the laser in order to compensate for tidal motions of the earth.

4.2 Optics

The primary function of test masses in the interferometer is to act as the mirrors that form the Fabry-Perot cavities in each of the interferometer arms. As such, both the optical and mechanical loss of these mirrors must be extremely low. The optical loss for the mirrors is measured at roughly a few 10s of parts per million. The test masses are made from high quality fused silica, which is specified to have zero inclusions. This material was selected because of its high quality bulk properties and because very high precision polishing had already been demonstrated.

As optical elements, the test mass mirrors must support roughly 100 round trips of light in 4 km long cavities without significant energy loss or mode distortion. In order to meet these demanding requirements there has been an extensive optical development program within LIGO[4]. This program was intended to seek out and engage vendors that could provide repeatable optical polish with surface deviations on the order of 0.8 nm rms (~λ/800 at 632 nm) over an 80mm diameter, after excluding errors due to power and astigmatism. The program also included development work that focused on the deposition of extremely uniform coatings. The current state of the technology is that the Commonwealth Scientific and Industrial Research Organization (CSIRO) in Australia and General Optics in the US are consistently delivering optics with an rms polish of ~0.5 nm rms over 80mm and ~1 nm rms over 200mm. This is the equivalent of λ/1200 measured over 80mm at 632 nm. The peak to valley uniformity of the coatings deposited by Research Electro Optics (REO) in the USA was found to be ~ 0.5%[5].

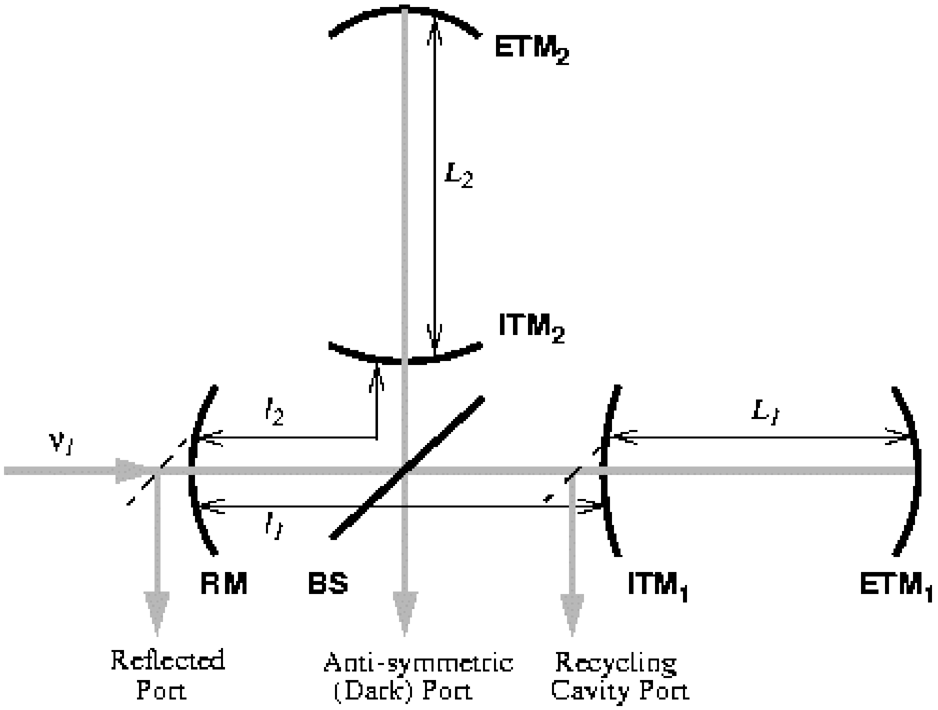

Figure 2: The layout of the LIGO interferometer is that of an enhanced Michelson. Three output ports are used in alignment sensing and control.

The mechanical properties of the masses are important because the room temperature vibrations of the atoms within the glass excite the internal modes of the test mass. Most of these vibrations will average out during the cavity storage time. However, the lowest frequency internal modes come close to overlapping the frequency band of the gravitational waves that LIGO is designed to detect. This problem is addressed by using masses with high quality factors (Q). When the mass has a high Q, the vibrations are constrained within a narrow frequency band, outside the LIGO detection frequency. The wires that suspend the test masses are also subject to the same considerations and can contribute to the thermal noise of the system.

The specific properties of the LIGO optics were developed using a computer model of a recycled Michelson interferometer with Fabry-Perot arms. This computer model uses FFT-based optical propagation. It includes surface and bulk maps of each of the six core optics. Using these parameters as fixed inputs the code then solves for the various interferometer signals and combines them to model the sensitivity to the gravitational wave signal. This code was used extensively in defining the radii of curvature and tolerance of the test masses as well as the sensitivity of the detector signal to surface and bulk optical quality. Some of the specifications for the interferometer optics are summarized in Table 1.

Physical Property	**Test Mass** **End (ETM)**	 **Input (ITM)**	**Beam Splitter**	**Recycling Mirror**
Diameter of Optic (mm)	250	250	250	250
Thickness of Optic (mm)	100	100	40	100
1 ppm intensity diameter (mm)	240	191	302	192
Lowest internal mode (kHz)	6.79	6.79	3.58	6.79
Mass of optic (kg)	10.7	10.7	4.2	10.7
Side 1 Radius of curvature (m)	7400	14570	infinite	14900
Rms surface error over 80mm diameter (nm)	< 0.8	< 0.8	< 1.6	< 1.6
Rms microroughness over 80mm diameter (nm)	0.2	0.2	0.4	0.4
Surface one Coating Transmission	< 20 ppm	3%	50%	3%

Table 1: Summary of LIGO optic specifications

4.3 Isolation

A single strand of steel wire suspends each interferometer optic. This leaves the optic/test mass free to move under the influence of a gravitational wave. Because of thermal noise considerations the wires are chosen such that the test mass loads the wire to one-half of its breaking strength. A grooved glass rod is used to minimize rubbing at the tangential point where the wire meets the glass. To apply small forces to the mirror for alignment and control, magnets are glued to aluminum standoffs, which are then glued to the mirror. Sensor/actuator heads are fitted to the suspension cage to detect the positions of these magnets. These heads use magnetic fields generated by a coil in order to push or pull on the magnets. The suspension cage itself is a high Q structure formed by welding machined stainless steel tubes and plates. As in the case of the test masses mentioned above, the resonance frequency of all sources must be pushed outside the gravitational wave band because of thermal noise considerations.

Each mirror suspension cage is placed on a four-layer seismic isolation stack; each stack occupies its own chamber. The chambers allow access for installation of the optics, baffles and the diagnostics that are required inside the vacuum. The chambers also have window ports that allow some beams to exit the vacuum system for analysis. The vacuum chambers are connected by a ~1m-diameter beam tube. The tubes contain optical baffles that are placed within the tubes for the purpose of trapping any scattered light so that it does not backscatter into the interferometer.

4.4 Controls

The interferometer is under the control of a sophisticated active alignment and length control system. Optimization of the interferometer involves a many dimensional control matrix that includes four lengths, three groups of frequency modulated light and their relative frequency shifts.

The servo system for the interferometer is designed to hold the length of the cavities on a fixed fringe such that the recombined light from the Fabry-Perot cavities cancels at the anti-symmetric port. Figure 2 shows the detection ports of the interferometer. The output signal from the control system, which is derived from the anti-symmetric port, is proportional to the differential strain in the two arms and therefore proportional to the strength and frequency of the gravitational wave. The four degrees of freedom for the length of the interferometer are:

- Differential lengths of the Fabry-Perot cavities (the gravitational wave signal) L_1-L_2
- Common motion of the cavities L_1+L_2
- Differential lengths of the near Michelson arms, formed by back reflection off of the input test masses (ITM) l_1-l_2
- Common motion of the near Michelson arms l_1+l_2

To maintain the best signal sensitivity these length combinations must be controlled to within 10^{-11} to 10^{-13} meters of the optimal operating point[6]. To sense the various lengths, phase modulation at a frequency of 20 – 30 MHz is applied to the input light using a Pockels cell. Consequently, the light is split into three frequencies, a carrier and two low amplitude side bands. The carrier frequency is resonant in the recycling cavity and in the Fabry-Perot cavities. The carrier side bands are resonant only in the recycling cavity. The signals are demodulated at three different ports to give error signals for each of the four lengths. A sample length control matrix is shown in Table 2.

Port/Signal	**L_1+L_2**	**l_1+l_2**	**L_1-L_2**	**l_1-l_2**
Reflection	-62000	-560	0	0
Recycling Cavity P.O.	5200000	17000	0	0
Anti-symmetric	0	0	23000	180
Reflection	0	0	0	19
Recycling Cavity P.O.	0	0	0	4900

Table 2: A sample control matrix in units of wavelength shows the sensitivity of the signal output at three monitoring ports to changes in the different lengths of the interferometer cavities. L_1–L_2 is the gravitational wave signal.

Angular alignment of the interferometer is also critical to maintaining optimum detector sensitivity. A proper alignment maximizes the power circulating in the arm cavities and minimizes the lost power that can couple into technical noise. The six core optics of the interferometer introduce twelve angular degrees of freedom into the control system, pitch and yaw from each optic.

The alignment of the interferometer is controlled by monitoring the signals from several wavefront sensors. A wavefront sensor is a multi-element photodetector, which can spatially resolve amplitude modulation that is present at the phase modulation frequency. Alignment errors produce spatial amplitude modulation and are detected in the frequency-demodulated signals from the wavefront sensors. The misalignment signal is mostly caused by interference of the TEM_{00} mode of the carrier and the misalignment-generated TEM_{10} and TEM_{01} modes of the sidebands. Since the interference signal depends on the Gouy phase angle between the TEM_{00} and TEM_{10} modes, the longitudinal position of the detector along the optic axis is also relevant. This means that angular misalignments of longitudinally separated optics can be distinguished by placing wavefront sensors at different positions along the optical path[6]. The three interferometer signal ports are shown in Figure 2. The signals from these ports are used to extract the alignment status and are also used in length sensing.

4.5 The 40m interferometer at Caltech

The 40 meter gravitational wave detector has been operating in various configurations since 1982[2]. This interferometer serves as a test bed for development of LIGO and has greatly enhanced our understanding of the operation of a gravitational wave detector. The evolution of the 40 m noise spectrum is shown in Figure 3. The improvements in system performance can be attributed to better control of measurement and control noise, and increase in the optical power delivered into the cavities. The most significant improvement however, is the in the displacement sensitivity achieved by redesigning the seismic isolation stacks to give approximately two orders of magnitude lower transmission at ~100 Hz. At frequencies between 500 and 1000 Hz, the Mark I interferometer was limited by mirror surface distortions due to thermal noise in the test masses. The new mirrors have Qs that are two orders of magnitude higher, and result in a decrease of nearly an order of magnitude in contribution to the thermal noise. The interferometer is shot noise limited at frequencies above one kHz.

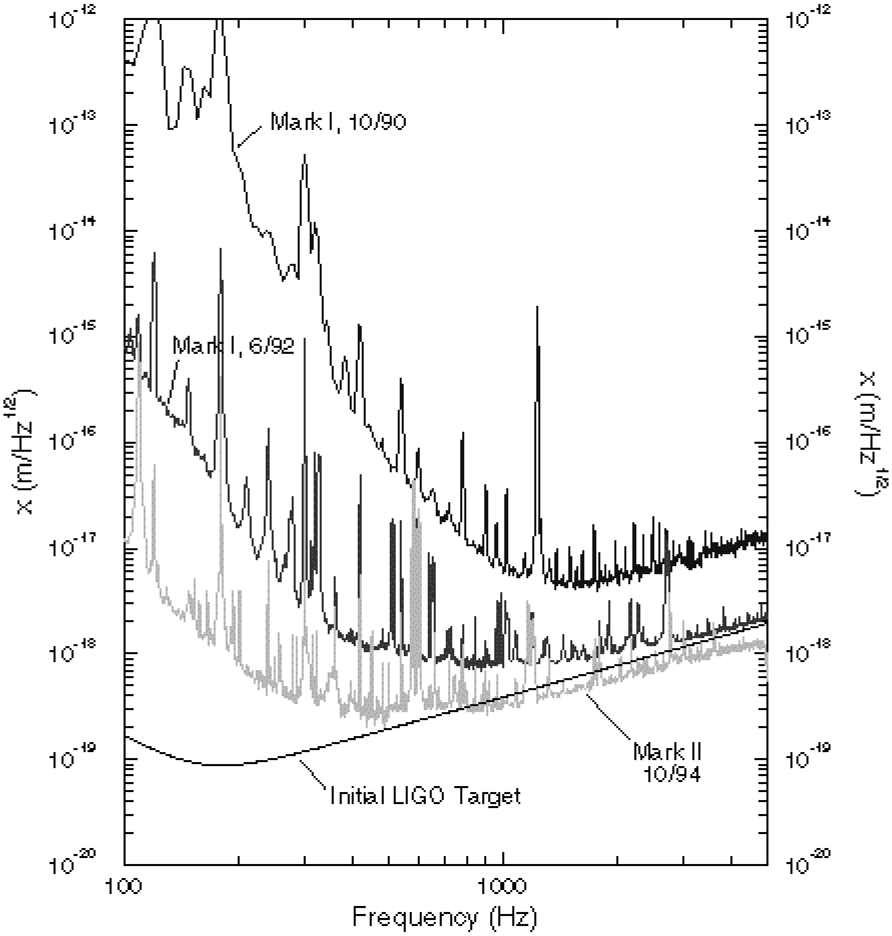

Figure 3: Improvements in the noise spectrum of the 40 meter interferometer are referenced to the initial target sensitivity of LIGO.

5. INTERNATIONAL COLLABORATION

A confident detection of gravitational waves can be made by LIGO alone. However, to obtain the maximum scientific benefit, an international collaboration has been formed that involves coordination with several gravitational wave observatories worldwide. The 3 km gravitational wave detector of VIRGO is expected to be operational in Pisa, Italy soon after LIGO. VIRGO is designed and built by collaboration among several groups in Italy and France. A gravitational wave detector, GEO600, is currently being constructed in a German and British collaboration. This is a 600 meter detector located in Hanover, Germany. There are several other efforts in various stages of progress throughout the world, namely TAMA in Japan and AIGO in Australia. Most of these detectors are based on an enhanced Michelson interferometer geometry.

6. SUMMARY

The facilities are in place at both of the LIGO observatories. In all, these facilities include several buildings, 33 enormous vacuum chambers and 16 km of welded, evacuated beam tube. All of this is laid out using precision global positioning. At Hanford, which has the first installation—the 2 km interferometer, the laser is in place and the interferometer optics are currently being suspended. At Louisiana—the first 4 km installation, the laser is being commissioned and input optics are being installed.

Plans are well underway to begin implementing “advanced LIGO” designs on the 40 meter interferometer at Caltech. The current LIGO design is cast, but there are plans to implement incremental improvements in many areas. For instance new materials, such as sapphire, are being evaluated for suitability as test masses. New suspension designs, such as a double pendulum, or even a glass fiber suspension are currently under investigation. Designs for a 100 Watt laser, based on slab geometry are being attempted by collaborators at Stanford and at the University of Adelaide.

One future vision for gravitational wave detection is a space-based interferometer. This work is done under the Laser Interferometer Space Antenna (LISA) program. The primary objective of this mission is to detect and observe gravitational waves from massive black holes and galactic binary stars in the frequency range 10-4 to 10-1 Hz. Useful measurements in this frequency range cannot be made on the ground because of the large background of local seismic noise.

The LISA concept now consists of six identical spacecraft forming an equilateral triangle in space with two closely spaced (200 km) ‘near' spacecraft at each vertex. The lasers of both spacecraft are phase-locked together, thus behaving effectively as a single laser. Each of the two neighboring spacecraft tracks a spacecraft at each of the other two distant vertices. Two central spacecraft correspond to the central mirror of a Michelson interferometer, while the two distant spacecraft correspond to the two end mirrors. When a gravity wave passes through the system, it causes a strain distortion of space, which will be detected by measuring the fluctuations in separation between proof masses inside the spacecraft. The separated masses constitute the arms of the interferometer. The measurements are performed by optical interferometry that determines the phase shift of laser light transmitted between ‘free-floating' test masses.

It is now 40 years since Weber began his development of gravitational wave detectors[7] and 30 years since Forward[8] and Weiss[9] initiated work on interferometric detectors. Since then, hundreds of experimental physicists have worked to improve the sensitivities of these instruments. These interferometers may detect the first gravitational waves in 2002 or soon thereafter, thus opening a rich new window on the Universe.

ACKNOWLEDGEMENTS

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