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<b>Principles of calculating the dynamical response of misaligned complex resonant optical interferometers</b>		
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# Principles of calculating the dynamical response of misaligned complex resonant optical interferometers

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In the long baseline laser interferometers for measuring gravitational waves that are now under construction, understanding the dynamical response to small distortions such as angular alignment fluctuations presents a unique challenge. These interferometers comprise multiple coupled optical resonators with light storage times approaching 100 msec. We present a basic formalism to calculate the frequency dependence of periodic variations in angular alignment and longitudinal displacement of the resonator mirrors. The electromagnetic field is decomposed into a superposition of higher-order spatial modes, Fourier frequency components and polarization states. Alignment fluctuations and length variations of free space propagation are represented by matrix operators that act on the multi-component state vectors of the field.

*Key words:* Gravitational-wave observatories, laser interferometer, misalignment of Gaussian beams, dynamical response of optical resonators, LIGO, mode decomposition, optical cavities.

## I. INTRODUCTION

A new generation of ground-based laser interferometric gravitational wave detectors with arm lengths of up to 4 km is currently under construction [1–4]. Their sensitivity to relative length changes is expected to be  $\Delta L/L \sim 10^{-21}$  rms integrated over a 1 kHz bandwidth centered in the region of minimum noise. To achieve such high sensitivity these systems implement complicated variations of a Michelson interferometer with multiple coupled optical resonators. The long baselines and coupled cavities of the optical resonators yield light storage times of order 0.5 msec to 100 msec. To maintain resonance conditions, the interferometer lengths must be tuned to within a small fraction of the cavity line width. To further ensure maximum sensitivity to gravitational waves, the interferometer mirrors must be held in precise angular alignment with respect to the incoming laser beam [5]. To implement a control system to keep the interferometer lengths closely tuned to resonance and to accurately infer the sensitivity to gravitational waves, the frequency response of the interferometer to changes in resonant lengths must be known. Similarly, understanding the interferometer response to angular misalignment is necessary to enable an alignment control system.

In the past, two separate but mathematically similar formalisms have been independently developed: one describes the dynamical behavior of slightly off-resonance interferometer fields [6,7]; and the other is useful for calculation of the effects of angular misalignment [8–12]. Both approaches are based on decomposing the laser field into a multi-component vector. In the dynamical response formalism the vector components represent different frequencies of the laser light, while in the angular misalignment formalism they represent different spatial modes of the field. In both approaches beam propagation and distortion effects are expressed as operators acting on these field vectors, making it possible to simulate complex resonant optical systems by evaluating relatively straightforward matrix equations.

In this paper we combine the two formalisms by defining a vector space which includes both the frequency and the spatial mode components of the field. For completeness we include the light polarization explicitly. Using a triangular ring cavity as a simple example, we derive the operators for propagation and reflection when both the resonant lengths and mirror angles are dithered at a given frequency (Section II). We then calculate the frequency response of the optical system in conjunction with a heterodyne detection scheme (Section III) and in presence of noise terms on the input light (Section IV). Finally, we provide a recipe for building more complex optical systems such as coupled Fabry-Perot cavities in the arms of a Michelson interferometer (Section V). We emphasize the generality of the model as an analytic tool to investigate the frequency dependence of distortion effects in any complex optical system—in particular coupled cavity systems.

## II. BASIC FORMALISM

It is useful to first study a relatively simple optical system, a triangular ring cavity, shown in Fig.1. The laser field inside the cavity,  $E_{\text{cav}}$ , can be expressed as the sum of the input field,  $E_{\text{in}}$ , and the field circulating in the cavity (due to the previous round trip),  $E'_{\text{cav}}$ .

$$E_{\text{cav}} = G_{\text{rt}} E'_{\text{cav}} + T_1 E_{\text{in}} \quad (1)$$

where  $G_{\text{rt}}$  denotes an operator which propagates the laser field by one round-trip and where  $T_1$  describes the transmission through the input mirror,  $M_1$ . In the steady state, the cavity resonance condition is satisfied when  $E_{\text{cav}} = E'_{\text{cav}}$ . It is important to recognize that this condition describes not only the static case, but also situations with periodic variations by using field vectors consisting of multiple frequency components.

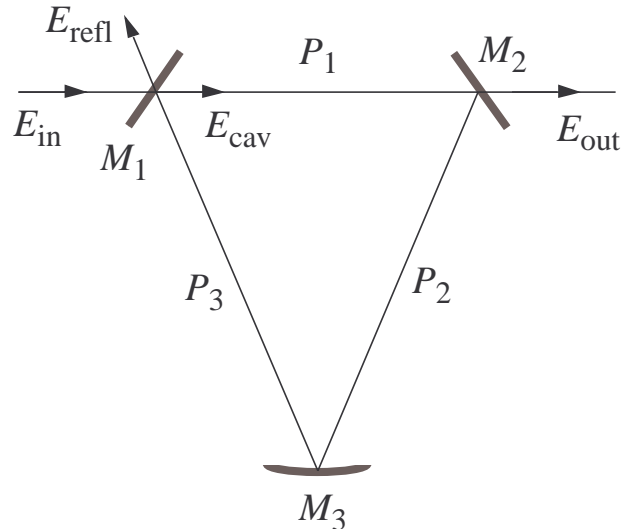


FIG. 1. Triangular Fabry-Perot cavity.

In the paraxial approximation the electromagnetic field vector of a light beam can be expressed as a superposition of orthonormal Gaussian modes [13]. If the time dependence of the field is periodic at frequency  $f$ , one can write the electric field in terms of Fourier components.

$$\vec{E}(x, y, z, t) = \sum_{mn} \sum_r \sum_p a_{mnrp} \exp[i(\omega_0 + r\omega)t] \times U_{mn}(x, y, z) \vec{e}_p \quad (2)$$

where  $\omega_0$  is the angular frequency of the laser light,  $\omega = 2\pi f$  is the fundamental angular frequency of the variation and  $\vec{e}_1$  and  $\vec{e}_2$  are the two transverse polarizations in  $x$  and  $y$ , respectively. Naturally, the  $U_{mn}(x, y, z)$  are chosen to represent the eigenmodes of an optical resonator.

For instance, a Hermite-Gaussian beam [13] can be described by

$$\begin{aligned}
U_m(x, z) &= \left(\frac{2}{\pi}\right)^{\frac{1}{4}} \left(\frac{1}{2^m m! w(z)}\right)^{\frac{1}{2}} H_m \left[ \frac{\sqrt{2} x}{w(z)} \right] \\
&\times \exp \left[ -x^2 \left( \frac{1}{w(z)^2} + \frac{ik_0}{2R(z)} \right) \right] \\
&\times \exp \left[ i \left( m + \frac{1}{2} \right) \eta(z) \right] \quad (3)
\end{aligned}$$

where the  $z$ -axis points in the beam propagation direction and where  $\eta(z)$ ,  $w(z)$ , and  $R(z)$  are the mode-dependent Gouy phase shift, the spot size, and the curvature of the phase front at position  $z$ , respectively,

$$\begin{aligned}
\eta(z) &= \tan^{-1} \left( \frac{z}{z_0} \right), \quad w(z) = w_0 \sqrt{1 + \left( \frac{z}{z_0} \right)^2}, \\
R(z) &= z + \frac{z_0^2}{z}, \quad (4)
\end{aligned}$$

and the Rayleigh length,  $z_0$ , is given by  $z_0 = \pi w_0^2 / \lambda$ , with  $w_0$  the waist size.  $H_m(x)$  is the Hermite polynomial of order  $m$ . In two dimensions the Hermite-Gaussian modes are given by

$$U_{mn}(x, y, z) = U_m(x, z) U_n(y, z). \quad (5)$$

In its most general form the quantity  $a_{mnrp}$  of Eq. (2) represents a tensor of rank four. The indices  $m, n$  refer to the spatial mode component, while the index  $r$  refers to the Fourier frequency component and  $p$  refers to the polarization state of the electric field. Since in most practical applications an operator acting on  $a_{mnrp}$  will contract over all four indices simultaneously—i.e., it will sum over all spatial modes, frequency components and polarization states—one can map  $a_{mnrp}$  into a one dimensional vector space. Assuming that only a finite number of frequency components,  $n_f$ , are considered,  $-\frac{n_f-1}{2} \leq r \leq \frac{n_f-1}{2}$ , one can write, for example,  $a_{mnrp} \rightarrow a_i$ :

$$i = 2n_f \left( \frac{(m+n)(m+n+1)}{2} + n \right) + a(r) + p \quad (6)$$

with

$$a(r) = \begin{cases} r & r \geq 0 \\ r + n_f & r < 0. \end{cases} \quad (7)$$

This convention allows to use standard linear algebra packages to perform numerical calculations; *Mathematica*<sup>TM</sup> was used to evaluate the examples in this paper. Our goal is to derive matrix operators acting on this vector space, which account for free space propagation and reflection from a mirror. We can then calculate the reflected and transmitted fields,  $E_{\text{refl}}$  and  $E_{\text{out}}$ , respectively, by solving simple matrix equations.

Using the operators  $P$  and  $M$  in Eq. (1) for propagation and reflection from a mirror surface, the round trip operator can be written as

$$G_{\text{rt}} = M_1 R P_3 M_3 P_2 M_2 P_1. \quad (8)$$

We follow the convention that the  $z$  axis is always defined along the beam direction, and the  $x$  and  $y$  axes are transverse and orthogonal. In general, the  $x$  and  $y$  axes preserve the orientation of the beam image after reflection from a mirror surface, i.e., the coordinate system changes between right-handed and left-handed with every reflection. However, the  $x$  and  $y$  axes must align again after one round trip of the ring cavity. In general, both a rotation of the beam (nonplanar configuration) and a parity operation (odd number of reflections) may occur (see Appendix A 1); this is represented by the operator  $R$  in Eq. 8. For a triangular cavity, however, only a parity operation which flips the beam about the vertical axis is necessary. To form a stable cavity at least one of the mirrors must be curved. In order to neglect astigmatic effects in a triangular cavity we assume that the length between  $M_1$  and  $M_2$  is much shorter than the other two mirror separations (see Fig.1).

Propagation in free space along the  $z$ -axis adds a term due to the Gouy phase shift,  $\exp[i(m+n+1)\eta]$ , which depends only on the spatial mode order and does not affect the polarization. Since a change of length affects neither the modal decomposition nor the polarization of the field, one can write

$$P_{mnrp,klpq} = P_{mn,kl}^M \otimes P_{rs}^F \otimes P_{pq}^P \quad (9)$$

where  $P^M$ ,  $P^F$  and  $P^P$  are operators acting on the modal space, the frequency space and the polarization states, respectively. The outer product “ $\otimes$ ” is used to combine operators which act on subspaces independently. For instance, in the above equation the modal operator  $P^M$  is applied to all frequency components and polarization states identically. Following Ref. [12] one can write beam propagation as

$$P_{mn,kl}^M(\eta) = \delta_{mk} \delta_{nl} \exp[i(m+n+1)\eta] \quad (10)$$

with  $\eta = \eta(z_2) - \eta(z_1)$ . The propagator  $P_{pq}^P$  accounting for polarization effects is unity. Assuming that the distance  $l(t) = z_2 - z_1 + \Delta l(t)$  is modulated by a small periodic signal  $\Delta l(t)$  at frequency  $f$ , one can write the Fourier series as

$$l(t) = \sum_{\rho} l_{\rho} \exp[i\rho\omega t] \quad (11)$$

with  $l_0 = z_2 - z_1$ . The propagator  $P_{rs}^F$  then becomes

$$\begin{aligned}
P_{rs}^F(l) &= \langle r | \exp \left[ -i \sum_{\rho} (k_0 + \frac{\rho\omega}{c}) l_{\rho} \exp[i\rho\omega t] \right] | s \rangle \\
&= \exp[-ik_0(z_2 - z_1)] \\
&\times \langle r | \exp \left[ -i \sum_{\alpha\beta} \langle \alpha | L_{\alpha\beta} \langle \beta | \right] | s \rangle \quad (12)
\end{aligned}$$

where  $k_0$  denotes the wave number of the laser light; the bra-ket product is defined as the integration in time over a full period; and  $\langle r |$  and  $| s \rangle$  are the Fourier basis vectors,  $\exp[-ir\omega t]$  and  $\exp[is\omega t]$ , respectively. Neglecting terms

of order  $l_\rho\omega/c$  ( $\rho \neq 0$ ) and using Eq. (11),  $L_{\alpha\beta}$  in the exponent of Eq. (12) reduces to

$$L_{\alpha\beta} \simeq \alpha \frac{\omega}{c} (z_2 - z_1) \delta_{\alpha\beta} + k_0 \sum_{\rho \neq 0} l_\rho \delta_{\alpha,\beta+\rho}. \quad (13)$$

Similarly, one can derive the mirror operator  $M_{mnrp,klqs}$ . Again, the polarization is an independent degree-of-freedom and can be treated separately. One can write the reflection from a mirror surface as

$$M_{mnrp,klqs} = M_{mnr,kl}^{MF} \otimes M_{pq}^P, \quad (14)$$

$$M_{mnr,kl}^{MF} = \langle mnr | \exp \left[ -2i \sum_{\alpha\beta\rho,\gamma\delta\sigma} |\alpha\beta\rho \rangle T_{\alpha\beta,\gamma\delta}^M \otimes T_{\rho\sigma}^F \langle \gamma\delta\sigma | \right] | kls \rangle \quad (15)$$

where the modal distortion operator  $T^M$  is assumed to affect all frequency components the same way. The light frequency is of order  $10^{14}$  Hz while physical disturbances of the mirrors and, therefore, the frequencies of modal distortions are typically in the audio band. Most reflective surfaces do not distinguish such small frequency variations of the laser light. The case where the mirror operator in Eq. (14) describes alignment fluctuations is, however, of special interest. If the mirror is dithered in angle about the  $y$  axis by  $\theta(t) = \sum_\tau \theta_\tau \exp[i\tau\omega t]$ , one obtains in the Hermite-Gaussian basis [12]

$$T_{\alpha\beta,\gamma\delta}^M = \frac{1}{\sqrt{2}} \delta_{\beta\delta} (\sqrt{\gamma} \delta_{\alpha,\gamma-1} + \sqrt{\alpha} \delta_{\alpha,\gamma+1}), \quad (16)$$

$$T_{\rho\sigma}^F = \sqrt{2} \sum_\tau \Theta_\tau \delta_{\rho,\sigma+\tau} \quad (17)$$

where  $\Theta_\tau = \theta_\tau \pi w(z)/\lambda$  denotes the normalized misalignment angle. Neglecting wavefront distortion effects when passing through a mirror substrate the transmission operator becomes

$$T = \mathbf{1}^M \otimes \mathbf{1}^F \otimes T^P \quad (18)$$

with  $\mathbf{1}$  the unity operator. If the reflectivity is independent of polarization, the corresponding operator  $M^P$  is just the amplitude reflectivity,  $r$ , times the unity operator. Similarly, the transmission operator  $T^P$  is described by the amplitude transmission coefficient only, i.e.

$$M_{pq}^P = r \delta_{pq} \quad \text{and} \quad T_{pq}^P = it \delta_{pq} \quad (19)$$

(the general case is treated in Appendix A 2). The convention of using an imaginary transmission coefficient makes the transmission operator,  $T$ , the same in both directions; and it makes the distortion operator,  $M^\dagger$ , that describes the reflection from the opposite direction (rear side of mirror coating) the complex conjugated transpose of the matrix that describes reflection from the front surface (assuming the coating is perfect).

### III. DETECTION SCHEME

A photodetector at position  $z$  measures the light power integrated over its surface area  $\Omega$ , i.e.

$$P_{\text{det}} = \frac{\epsilon_0 c}{2} \iint_{\Omega} dx dy \vec{E}^*(x, y, z) \cdot \vec{E}(x, y, z) \quad (20)$$

where  $\epsilon_0$  and  $c$  are the free space permittivity and the speed of light, respectively. The shape of the photodetector, including segmentation in the case of a quadrant photodetector, e.g., is contained in the area,  $\Omega$ . Since the light power of Eq. (20) keeps track of frequency components, but integrates over spatial modes and sums over polarization states, one can write an operator equation for the detected light power at frequency  $\rho f$ :

$$P_{\text{det}}(\Omega, \rho) = \frac{\epsilon_0 c}{2} E_{klsq}^* D_{klsq,mnrp}(\Omega, \rho) E_{mnrp} \quad (21)$$

where the sum is taken over all indices. An ideal photodetector doesn't intermix frequency components, spatial modes and polarization states, and the operator  $D$  can be simplified to

$$D_{mnrp,klqs}(\Omega, \rho) = D_{mn,kl}^M \otimes D_{rs}^{F\rho} \otimes D_{pq}^P. \quad (22)$$

The operator  $D_{mn,kl}^M$ , which acts on the modal space and accounts for the shape of the detector, is given in Ref. [12]. The  $\rho$ -th frequency component,  $\rho f$ , is projected out by the operator,  $D_{rs}^{F\rho}$ , which selects the Fourier components,  $\langle r |$  and  $| s \rangle$ , such that  $s - r = \rho$ . Polarization states are added in quadrature, therefore,

$$D_{rs}^{F\rho} = \delta_{r,s-\rho} \quad \text{and} \quad D_{pq}^P = \delta_{pq}. \quad (23)$$

In a heterodyne detection scheme additional frequencies are added to the light. In the Pound-Drever-Hall reflection locking technique [14], for example, an electro-optic modulator is used to impose phase-modulation sidebands at radio frequencies (rf) on the light incident on a Fabry-Perot cavity. When it is slightly detuned from resonance, the cavity converts the phase modulation to amplitude modulation, and the light reflected from the cavity contains an rf signal which is proportional to the deviation from perfect resonance. There are two approaches to integrating these additional rf frequencies into the above formalism: (i) extending the frequency space or, (ii) calculating the electric field vectors of the output field separately (assuming there is no interaction between the different rf components). We use the latter approach, in which case the down-converted power signal can be written as

$$S(\Omega, \rho) = \frac{\epsilon_0 c}{2} E_{\text{CR}}^* D(\Omega, \rho) E_{\text{SB}-} + \frac{\epsilon_0 c}{2} E_{\text{SB}+}^* D(\Omega, \rho) E_{\text{CR}} \quad (24)$$

where  $E_{\text{CR}}$ ,  $E_{\text{SB}+}$  and  $E_{\text{SB}-}$  denote the electric field of the carrier, the upper sideband and the lower sideband, respectively. The above equation is very similar

to Eq. (21) with the difference that the down-conversion selects only the cross terms at the rf frequency.

In general,  $S(\Omega, \rho)$  is a complex quantity: the real part describes the in-phase (cosine) term while the imaginary part describes the quadrature-phase (sine) term of the rf signal. If one adds positive and negative audio frequencies, one obtains a periodic signal at frequency  $\rho f$ , which describes an ellipse in the complex phase plane:

$$S = S(\Omega, \rho) + S(\Omega, -\rho) \\ = \exp[i\varphi] \{a \cos(\rho\omega t + \phi) + ib \sin(\rho\omega t + \phi)\} \quad (25)$$

where  $a$  and  $b$  are the major and minor axes of the ellipse,  $\varphi$  gives the orientation of the major axis and  $\phi$  is the phase shift of the signal. In practice, the minor axis often vanishes, i.e.  $b = 0$ , and the signal can be described by its complex transfer function coefficient,  $a \exp[i\phi]$ , and its rf-phase,  $\varphi$ , only.

#### IV. INPUT BEAM NOISE

An important consideration in designing an interferometric gravitational wave detector is the coupling of noise on the input light to the gravitational wave read-out port. The effects of these noise sources can readily be calculated by introducing “noise” operators which add various noise terms to an otherwise pristine input beam. Important noise sources are the laser frequency noise, the laser amplitude noise, the oscillator phase noise, the oscillator amplitude noise and the input beam jitter. Oscillator phase and amplitude noise are added to the system by the rf sideband generating process which is done by an rf oscillator driving a Pockels cell. Angular input beam jitter can be simulated by using the mirror operator from Eq. (14) and applying it to the input beam with a dither amplitude half the angular beam jitter amplitude [5]. Lateral input beam jitter can be simulated by replacing the modal mirror operator above with the lateral shift operator from Ref. [12].

Since laser and oscillator noise affects neither the modal decomposition nor the polarization state of the input beam one can write their corresponding noise operators as

$$N = \mathbf{1}^M \otimes N^F \otimes \mathbf{1}^P. \quad (26)$$

Using the expressions

$$f(t) = f_0 + \sum_{\rho \neq 0} \frac{\rho\omega}{2\pi} \phi_\rho \exp[i\rho\omega t] \quad \text{and} \quad (27)$$

$$A(t) = A_0 \left( 1 + \sum_{\rho \neq 0} \frac{\Delta A_\rho}{A_0} \exp[i\rho\omega t] \right) \quad (28)$$

for the laser frequency and the laser amplitude, respectively, and following the derivations of Ref. [15] one obtains the laser frequency noise operator,  $N_F^F$ , and the laser amplitude noise operator,  $N_A^F$ ,

$$(N_F^F)_{rs} = \langle r | \exp \left[ -i \sum_{\alpha\beta} |\alpha\rangle (T_F^F)_{\alpha\beta} \langle \beta| \right] | s \rangle, \\ (T_F^F)_{rs} = i \sum_{\rho \neq 0} \phi_\rho \delta_{r,s+\rho}, \quad (29)$$

$$(N_A^F)_{rs} = \mathbf{1} + \sum_{\rho \neq 0} \frac{\Delta A_\rho}{A_0} \delta_{r,s+\rho}. \quad (30)$$

Representing the rf modulation by a multiplication factor,  $\exp[i\Gamma \cos \omega_m t]$ , with  $\Gamma$  the modulation depth and  $\omega_m$  the modulation frequency one can calculate the oscillator phase noise and the oscillator amplitude noise by replacing

$$\omega_m t \rightarrow \omega_m t + \sum_{\rho \neq 0} \varphi_\rho \exp[i\rho\omega t] \quad \text{and} \quad (31)$$

$$\Gamma \rightarrow \Gamma_0 \left( 1 + \sum_{\rho \neq 0} \frac{\Gamma_\rho}{\Gamma_0} \exp[i\rho\omega t] \right). \quad (32)$$

Assuming  $\varphi_\rho \ll 1$  and  $\Gamma_\rho \ll 1$  and again following Ref. [15] one obtains the oscillator noise operator,  $N_O^F$ ,

$$(N_O^F)_{rs} = \sum_{\rho \neq 0} \{ \Gamma_\rho \ln' J_x(\Gamma) + ix\varphi_\rho \} \delta_{r,s+\rho} \quad (33)$$

where  $J_x(\Gamma)$  denotes the Bessel function and with  $x = 0, 1$  and  $-1$  for the carrier, the upper rf sideband and the lower rf sideband, respectively. Since oscillator phase noise is also present on the local oscillator of the down-conversion, the inverse of the oscillator phase noise operator has to be applied to the output beam before calculating the down-converted signals. Most down-conversion circuits, however, implement an amplitude limiter on the local oscillator, so the oscillator amplitude noise operator has to be applied to the input beam only.

#### V. APPLICATION TO COMPLEX OPTICAL SYSTEMS

Returning to the example of the triangular cavity, we can solve Eq. (1) for the field inside the cavity in the stationary case:

$$E_{\text{cav}} = (\mathbf{1} - G_{rt})^{-1} T_1 E_{\text{in}} \quad (34)$$

and for the cavity transmitted and reflected fields:

$$E_{\text{out}} = T_2 P_1 (\mathbf{1} - G_{rt})^{-1} T_1 E_{\text{in}}, \quad (35)$$

$$E_{\text{refl}} = \left[ T_1 M_1^{-1} G_{rt} (\mathbf{1} - G_{rt})^{-1} T_1 + M_1^\dagger \right] E_{\text{in}}. \quad (36)$$

In practice, one is often interested in calculating the dynamical response of an optical system “on resonance”. The length of an optical resonator can be microscopically adjusted so that the power build-up reaches a maximum. If  $E_n$  is an eigenvector of the round-trip operator,  $G_{rt}$ ,

with eigenvalue,  $e_n$ , the power build-up for this eigenmode is proportional to

$$P_{\text{cav}} \propto \frac{1}{|1 - e_n \exp[ik_0 \Delta l]|^2} \quad (37)$$

where  $\Delta l$  is the required length adjustment. The power build-up is maximal for  $\Delta l = -\arg(e_n)/k_0$ . Often one chooses the eigenmode which most closely represents the spatial mode of the incident beam, usually a  $\text{TEM}_{00}$  mode (see Eq. 3).

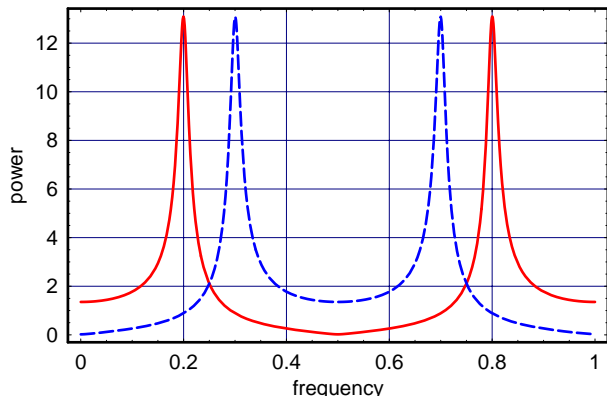


FIG. 2. Angular transfer function of a triangular Fabry-Perot cavity for horizontal (solid curve) and vertical misalignments. Shown is the power of the down-converted signal measured by a split-plane photodetector in the near field of the reflection port as function of the dither frequency (in units of the free-spectral-range) of mirror  $M_3$ . For parameters see text.

Fig.2 shows the response of the demodulated signal reflected from a ring cavity as a function of the frequency of the dither applied to the cavity mirror angle. This is referred to as an angular transfer function. Mirror  $M_3$  is dithered in angle and the signal is measured by a split-plane photodetector in the near field (close to the input mirror) of the cavity reflection port. At low dithering frequencies only the horizontal dither is detected in the near field (due to the necessary parity operation in a triangular cavity), whereas the vertical dither is most sensitively measured in the far field (at infinite distance from the input mirror). At higher frequencies both near and far field signals are significant and show peaks at the resonance frequencies for the  $\text{TEM}_{10}$  and  $\text{TEM}_{01}$  modes, respectively. These peaks can be understood as a resonant build-up of either  $\text{TEM}_{10}$  or  $\text{TEM}_{01}$  mode inside the cavity which happens at the frequencies where the additional phase shift from the free space propagation cancels the Gouy phase shift. When the power of one of these modes builds up inside the cavity, the amount of light leaking out of the cavity in reflection also increases, which in turn enhances the signal measured by the photodetector. It should be stressed that such a resonant build-up does not invalidate any of the assumptions which went into the formalism. In fact, the model works equally well on-resonance and off-resonance. The following param-

eters were used for the calculation:  $\lambda = 1.064$  nm (laser wavelength),  $l_1 = 0$  m,  $l_2 = l_3 = 6$  m (cavity lengths),  $\eta = 0.6\pi$  (round-trip Gouy phase shift),  $r = 0.97$  (amplitude reflectivity coefficient of mirror  $M_1$  and  $M_2$ , and  $f_{\text{rf}} = c/2/(l_2 + l_3)$  (rf modulation frequency). Mirror  $M_1$  and  $M_2$  were flat, whereas mirror  $M_3$  had a radius of curvature of  $\sim 9.2$  m which puts the beam waist right in between  $M_1$  and  $M_2$ . The input light was a purely vertically polarized  $\text{TEM}_{00}$  mode with a co-sinusoidal rf modulation. The photodetector at the reflection port was split into two half-planes which add with opposite sign. The detected output power was down-converted and normalized by the input power, the modulation index and the amplitude of the dither. The matrix notation of the operators used in the calculations are listed in Appendix A 3.

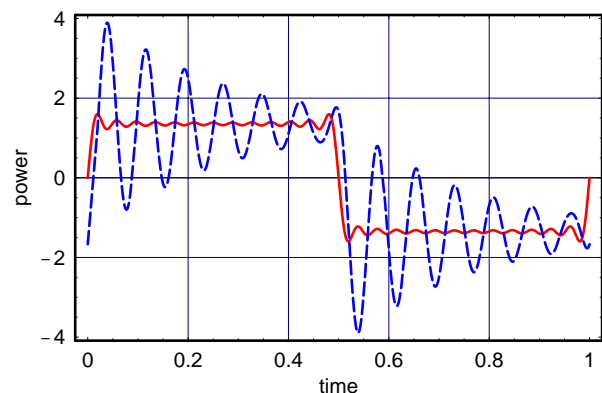


FIG. 3. Response of a triangular cavity to a square wave dither in angle. Shown is the power of the down-converted signal measured by a split-plane photodetector in the near field of the reflection port while mirror  $M_3$  is dithered by an approximation of a square wave signal. The frequency of the dither signal is 1 kHz (solid curve) and 384.947 kHz, respectively. The 1 kHz response is almost identical in shape to the applied dither signal, whereas for the 384.947 kHz dither one of the higher frequency components of the square wave becomes resonant. See text for parameters.

Fig.3 shows a simulation using 50 audio frequency components. The dither signal was an approximated square wave  $\frac{4}{\pi} \sum_{n=1}^{13} \frac{1}{2n-1} \sin [2\pi(2n-1)t]$ . Other parameters are the same as listed above.

Arbitrarily complex configurations such as a system of coupled optical resonators can be simulated by writing down the equivalent equations and using the operators of the previous sections to describe propagation, reflection and photodetection. This approach is similar to that of Ref. [12] and was used to build a model of the LIGO interferometer and calculate its angular transfer functions.

Using the equations for the power recycled Michelson interferometer with Fabry-Perot arm cavities in the arms from Ref. [12] and the optical parameters of the LIGO detector [5], the frequency dependence of the angular dithers were calculated. Fig.4 shows the alignment signals measured at the anti-symmetric port and the reflec-

tion port of the interferometer while dithering the end mirror of the in-line arm as a function of the dither frequency. The periodicity of the signal is determined by the free spectral range (fsr) of the LIGO arm cavity, which is about 37 kHz. The first resonance in the angular transfer function appears at 11.4 kHz, which is the fundamental frequency of the arm cavity transverse modes. The second peak in Fig.4 is the image of the 11.4 kHz resonance mirrored at the cavity fsr. At low frequencies, where the response functions are important for the automatic alignment system, there is no significant frequency dependence.

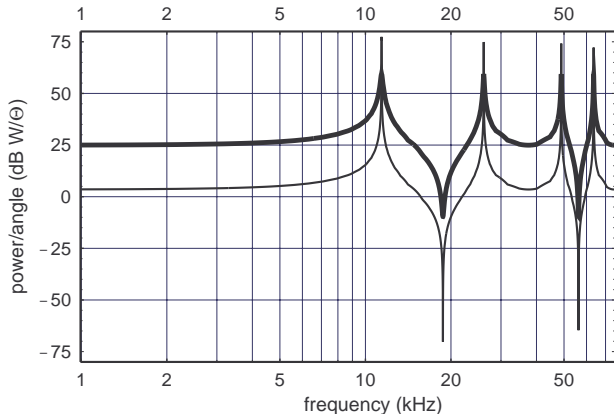


FIG. 4. Response of the LIGO interferometer to a dither of the rear mirror of the in-line arm cavity. Shown is the power of the down-converted signal per unit angle ( $\Theta \approx 10^{-5}$  rad) measured by a split-plane photodetector in the far field of the antisymmetric port (thick line) and in the far field of the reflection port, respectively. The reflection port signal is much narrower since it experiences a double resonance in both the arm cavity and the power recycling cavity; for higher frequencies the peak values slightly decrease indicating that the signal is shifted away from the exact resonance in the power recycling cavity. See text for parameters.

## VI. CONCLUSIONS

The model described in the present work is capable of simulating a wide set of problems encountered in designing an interferometric gravitational wave detector: (i) The calculation of displacement and angular sensitivity, (ii) the derivation of the frequency dependence of error signals used to lock and align these interferometers, and (iii) the study of technical noise sources such as the propagation of laser frequency noise through the system. Compared to methods which are based on FFTs [16–18] or full time domain simulations [19–21], this model is numerically economical because it gives explicit solutions for the laser fields in the interferometer. However, our approach cannot compete with these other methods if the distortions are very large (e.g. small apertures, large misalignment angles), or if one is interested in studying

transient phenomena. The techniques of this paper were used in the design of the LIGO detector.

Summarizing, this model is accurate and efficient if the distortions are small, periodic and can be expressed as a finite sum of the lowest order spatial eigenmodes of an optical resonator. It is also versatile in that it allows for building up of more complex optical systems with relative simplicity.

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## APPENDIX A: FORMULAS

### 1. Rotation and Parity Operator

Going through one round-trip of a ring cavity can require a physical rotation of the beam (non-planar configurations) as well as a parity operation (odd number of mirrors). A physical rotation of the beam will also rotate the polarization state accordingly. However, a rotation of the polarization state can be done independently of a physical rotation of the beam image. The most general rotation of a polarization state can be written as

$$R^P = \exp[-i\vec{a} \cdot \vec{\sigma}] = \mathbf{1} \cos |\vec{a}| - i \frac{\vec{a} \cdot \vec{\sigma}}{|\vec{a}|} \sin |\vec{a}| \quad (\text{A1})$$

with  $\vec{\sigma}$  the Pauli matrices

$$\vec{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \vec{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \vec{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (\text{A2})$$

and  $\vec{a}$  an arbitrary real vector. Any rotation (including a possible parity operation) can then be written as

$$R = (R^M F^M) \otimes \mathbf{1}^F \otimes (R^P F^P) \quad (\text{A3})$$

with  $F$  the parity operator which flips (mirrors) the beam along the  $y$ -axis. In the Hermite-Gaussian basis

$$F_{mn,kl}^M = (-1)^{mb} \delta_{mk} \delta_{nl}, \quad (\text{A4})$$

$$F^P = (-\sigma_3)^b \quad (\text{A5})$$

with  $b$  the number of mirror reflections. Starting with a rotation of the electric field vector

$$E(x \cos \alpha + y \sin \alpha, y \cos \alpha - x \sin \alpha) = R^M(\alpha) \cdot E(x, y, z) \quad (\text{A6})$$

one can expand the left-hand side in a Taylor series and obtains

$$R^M(\alpha) = \exp\left[i\alpha\left(y\frac{1}{i}\frac{d}{dx} - x\frac{1}{i}\frac{d}{dy}\right)\right]. \quad (\text{A7})$$

In the modal basis, this unitary operator can be written as

$$R_{mn,kl}^M(\alpha) = \langle mn | \exp\left[i\alpha\sum_{op,qr} |op\rangle T_{op,qr}^\alpha \langle qr| \right] |kl\rangle \quad (\text{A8})$$

with

$$T_{op,qr}^\alpha = \iint_{-\infty}^{\infty} dx dy U_o^\dagger(x, z) U_p^\dagger(y, z) \times \left[ \frac{y}{i} \frac{d}{dx} - \frac{x}{i} \frac{d}{dy} \right] U_q(x, z) U_r(y, z). \quad (\text{A9})$$

Using the recursion relations for the Hermite-Gaussian polynomials one can solve the integration:

$$T_{op,qr}^\alpha = \frac{\sqrt{pq}}{i} \delta_{o,q-1} \delta_{p,r+1} - \frac{\sqrt{or}}{i} \delta_{o,q+1} \delta_{p,r-1}. \quad (\text{A10})$$

The corresponding rotation of the polarization state is simply Eq. (A1) with  $a_2 = \alpha$  and  $a_1 = a_3 = 0$ .

## 2. Reflection and Transmission Coefficients

If a laser beam hits a surface at an angle other than normal, the reflection and transmission coefficients are in general different for perpendicular (s-polarization) and parallel (p-polarization) electric field components. If the  $x$ -axis is rotated by an angle  $\xi$  with respect to the direction of parallel polarization, the amplitude reflection operator  $M^P$  can be written as

$$M^P = \exp[i\vec{a}\vec{\sigma}] \begin{pmatrix} r_p & 0 \\ 0 & r_s \end{pmatrix} \exp[-i\vec{a}\vec{\sigma}] \quad (\text{A11})$$

with  $\vec{a} = (0, \xi, 0)$ . Similarly, the amplitude transmission operator  $T^P$  can be deduced from the above equation by replacing the perpendicular and parallel amplitude reflection coefficients,  $r_s$  and  $r_p$ , with their corresponding amplitude transmission coefficients,  $t_s$  and  $t_p$ , respectively.

## 3. Examples in Matrix Notation

When calculating the lowest order perturbations only, one can restrict the modal space to three dimensions (TEM<sub>00</sub>, TEM<sub>10</sub> and TEM<sub>01</sub>) and carry along only three audio frequency components (0,  $+f$  and  $-f$ ). This section lists the operators in the matrix notation for this finite vector space.

By using Eqs. (12) and (13) and assuming the propagation length is dithered by  $l(t) = l_0 + \delta l \cos \omega t$ , the free space propagator  $P$  becomes

$$P^F(l) = \exp[-ik_0 l_0] \times \exp\left[-i\begin{pmatrix} 0 & k_0 \delta l/2 & k_0 \delta l/2 \\ k_0 \delta l/2 & \omega l_0/c & 0 \\ k_0 \delta l/2 & 0 & -\omega l_0/c \end{pmatrix}\right],$$

$$P^M(\eta) = \begin{pmatrix} \exp[i\eta] & 0 & 0 \\ 0 & \exp[2i\eta] & 0 \\ 0 & 0 & \exp[2i\eta] \end{pmatrix}, \quad (\text{A12})$$

$$P^P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

neglecting terms of order  $\omega \delta l/c$ . Similarly, assuming that the mirror orientation is dithered by  $\theta(t) = \theta \cos \omega t$ , the mirror operators of Eqs. (14) to (17) can be written as

$$T^F = \frac{\Theta}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^M = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$M^P = \begin{pmatrix} r_p & 0 \\ 0 & r_s \end{pmatrix} \quad (\text{A13})$$

with  $\Theta = \theta \pi w(z)/\lambda$  and with  $r_p$  and  $r_s$  the amplitude reflection coefficients for p- and s-polarization, respectively. Using the relation of Eq. (6) one can map both the free space propagator and the mirror operators into a 18 dimensional vector space.

The rotation and parity operators can be expressed as (set  $\alpha = 0$  for a triangular cavity)

$$R^M = \exp\left[i\alpha\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}\right], \quad F^M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$R^P = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}, \quad F^P = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (\text{A14})$$

The operators for a circular photodetector,  $D_c$ , and a half-plane detector,  $D_h$ , (split along the  $y$ -axis, segments added with opposite sign) read

$$D^{F(+f)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad D^{F(-f)} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$D^P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (\text{A15})$$

$$D_c^M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D_h^M = \sqrt{\frac{2}{\pi}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

where operators without a subscript are independent of the shape of the detector. Assuming that the laser frequency noise can be expressed by  $f(t) = f_0 + \frac{\omega}{2\pi} \phi \cos \omega t$ , the laser amplitude noise by  $A(t) = A_0(1 + \Delta_A \cos \omega t)$ , the oscillator phase noise by  $\varphi(t) = \varphi \cos \omega t$  and the oscillator amplitude noise by  $\Gamma(t) = \Gamma(1 + \Delta_\Gamma \cos \omega t)$  one obtains

$$\begin{aligned}
N_F^F &= \begin{pmatrix} 1 & -\phi/2 & \phi/2 \\ \phi/2 & 1 & 0 \\ -\phi/2 & 0 & 1 \end{pmatrix}, \\
N_A^F &= \begin{pmatrix} 1 & \Delta_A/2 & \Delta_A/2 \\ \Delta_A/2 & 1 & 0 \\ \Delta_A/2 & 0 & 1 \end{pmatrix}, \\
N_{O\varphi}^F &= \begin{pmatrix} 1 & ix\varphi/2 & ix\varphi/2 \\ ix\varphi/2 & 1 & 0 \\ ix\varphi/2 & 0 & 1 \end{pmatrix}, \\
N_{OA}^F &= \begin{pmatrix} 1 & |x|\Delta_\Gamma/2 & |x|\Delta_\Gamma/2 \\ |x|\Delta_\Gamma/2 & 1 & 0 \\ |x|\Delta_\Gamma/2 & 0 & 1 \end{pmatrix}
\end{aligned} \tag{A16}$$

with  $\Gamma \ll 1$  and with  $x = 0, 1$  and  $-1$  for the carrier, the upper rf sideband and the lower rf sideband, respectively.

For computing the operators for the input beam jitter one can use the ansatz from Eqs. (16) and (17). If the input beam jitter is parameterized at a distance  $z$  away from the beam waist position by  $\alpha(t) = \alpha \cos \omega t$  (tilt around the  $y$ -axis) and  $\Delta x(t) = \Delta x \cos \omega t$  (displacement along the  $x$ -axis), the corresponding operators become

$$\begin{aligned}
T_\alpha^F &= \frac{\alpha \pi w(z)}{\sqrt{8\lambda}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T_\alpha^M = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
T_{\Delta x}^F &= -\frac{\Delta x}{\sqrt{8}w(z)} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
T_{\Delta x}^M &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & t^* & 0 \\ t & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned} \tag{A17}$$

with  $t = i - \frac{z}{z_0}$ .

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