The central density of a neutron star is unaffected by a binary companion at linear order in μ/R

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ABSTRACT

Recent numerical work by Wilson, Mathews, and Marronetti [J. R. Wilson, G. J. Mathews and P. Marronetti, Phys. Rev. D **54**, 1317 (1996)] on the coalescence of massive binary neutron stars shows a striking instability as the stars come close together: Each star's central density increases by an amount proportional to 1/(orbital radius). This overwhelms any stabilizing effects of tidal coupling [which are proportional to $1/(\text{orbital radius})^6$] and causes the stars to collapse before they merge. Since the claimed increase of density scales with the stars' mass, it should also show up in a perturbation limit where a point particle of mass μ orbits a neutron star. We prove analytically that this does *not* happen; the neutron star's central density is *unaffected* by the companion's presence to linear order in μ/R. We show, further, that the density increase observed by Wilson et. al. could arise as a consequence of not faithfully maintaining boundary conditions.

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Wilson, Mathews, and Marronetti (WMM) [1] have proposed a method of approximating the fully General Relativistic analysis of binary neutron star coalescence. The essence of their scheme is to choose a simple form of the spacetime metric (one in which the spatial three slices are conformally flat), and solve the constraint equations of General Relativity (GR) for some initial matter configuration. They evolve only the fluid equations forward in time until the fluid reaches a quasi-equilibrium configuration, then solve the constraint equations again for the new matter configuration and iterate until a quasi-equilibrium solution to the combined Einstein-fluid equations is found. Their method makes 3-dimensional simulations of such systems more tractable by reducing the computational requirements.

These simulations yield an extremely surprising result: neutron stars that are close to the maximum allowed mass are "crushed" into black holes long before the neutron stars coalesce. They claim that the origin of this effect is a non-linear gravitational interaction due to the companion's presence that strengthens the gravitational potential of each star. Consider a binary star system—star-A has mass M_A, and star-B has mass M_B. WMM claim that non-linear interactions cause the potential at star-A to be increased by a term that scales as M_B/R (where R is the orbital separation). This, in turn, increases the internal energy and density of star-A by terms that scale as M_B/R. If star-A happens to be marginally stable in isolation, the effect is sufficient to push it over the edge, causing a catastrophic collapse to a black hole.

The scaling law claimed by WMM is precisely what one would expect *if this effect were due to a post-1-Newtonian* enhancement of the gravitational interaction. Motivated by this observation, Wiseman has recently done a careful analysis of the effect that a binary companion has on a fluid star, using the first post-Newtonian approximation to GR [2]; he finds no change to either the central energy density or the angle averaged proper radius of the star at this order. Wiseman's calculation does not rule out completely a star-crushing effect, but does show that it is not evident at post-1-Newtonian order in GR.

Suppose for a moment that the WMM effect is a property of neutron star binaries in GR, and that it scales as M_B/R at star-A. Clearly, it should also be apparent in the limit that we shrink star-B down to a point particle of mass $\mu \ll M_A \equiv M$. In this limit, the exact solution of the Einstein field equations describing a binary neutron star system can be approximated by a perturbative expansion in μ/R about the solution for an isolated star. We write the metric as $g_{\alpha\beta} = g^0_{\alpha\beta} + \epsilon h_{\alpha\beta} + O(\epsilon^2)$, where the superscript 0 indicates the background metric, and we have introduced an order counting parameter ϵ with the formal value unity. Quantities multiplied by ϵ scale linearly with μ/R, quantities multiplied by ϵ^2 scale with $(\mu/R)^2$, *etc.* In what follows, we ruthlessly discard all terms of order ϵ^2, constructing an argument that is valid only to linear order in μ/R.

The neutron star material is considered to be perfect fluid with stress-tensor

$$T_{\alpha\beta} = P\, g_{\alpha\beta} + (P+\rho) u_\alpha u_\beta \,. \tag{1}$$

This must be supplemented with an equation of state relating the pressure P and the energy density ρ. For concreteness, we assume a polytropic form $P = Kn^\Gamma$ where K and Γ are constants, and n is the fluid's baryon density. The energy density ρ is directly related to n by the first law of thermodynamics; see Eqs. (3.2.6-7) of Ref. [4].

We take the background spacetime to be that of an isolated, spherical star with the line element

$$ds^2 = -e^{2\Phi(r)} dt^2 + \frac{dr^2}{[1-2m(r)/r]} + r^2 d\Omega^2. \tag{2}$$

Here $m(r)$ is the gravitational mass inside a sphere of radius r, and $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$. The combined Einstein-perfect fluid equations, generally referred to as the Oppenheimer-Volkoff (OV) equations (see for example Chapter 23 of Ref. [3]) are solved by demanding regularity of the origin $[m(0) = 0]$ and fixing the value of the central baryon density n_c. The radius of the star R^0_S is the coordinate radius at which the baryon density n^0 becomes zero. The central density n_c uniquely determines R^0_S, the total mass $M = m(R^0_S)$, and baryon mass M_b; this statement is equivalent to Theorem 7 of Ref. [5].

One can show using the polytropic equation of state and the OV equation for the pressure,

$$\frac{dP^0}{dr} = -\frac{(\rho^0 + P^0)[m(r) + 4\pi r^3 P^0]}{r[r-2m(r)]} \,, \tag{3}$$

that

$$\left.\frac{dn^0}{dr}\right|_{r\to 0} \to 0 \,. \tag{4}$$

Finally, the background geometry outside the star is described by the Schwarzschild solution with $m = M$ and $\exp(2\Phi) = 1 - 2M/r$ in Eq. (2).

The perturbing source is a single point particle of proper mass μ in a circular orbit at radius R. It is described by the stress-energy tensor [6]

$$T^{\alpha\beta} = \frac{\epsilon\mu}{R^2}\frac{v^\alpha v^\beta}{v^t}\delta(r-R)\delta(\cos\theta)\delta(\phi - \Omega t), \tag{5}$$

where $v^\alpha = (1-3M/R)^{-1/2}(1,0,0,\Omega)$ and $\Omega = \sqrt{M/R^3}$. The presence of this point "star" will alter the geometry and disturb the material in the central star, modifying the description of the spacetime and matter by terms of order ϵ. Linearizing $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$ and $T^{\alpha\beta}{}_{;\beta} = 0$ in ϵ, we find that the first-order perturbation equations separate by expanding the angular dependence in spherical harmonics, and the time dependence in Fourier modes. This is enough to address the issue of how the central density scales.

Consider the expansion of the baryon mass density. It may be written

$$n(r,\theta,\phi,t) = n^0(r) + \epsilon \sum_{l,m,\omega} \delta n_{lm\omega}(r) Y_{lm}(\theta,\phi) e^{i\omega t} \; . \quad (6)$$

An immediate consequence of Eq. (6) is that $\delta n_{lm\omega}(0) = 0$ for $l \geq 1$: if it were non-zero, the density would be multi-valued at $r = 0$. Thus, only the monopole could affect the central density if the center of the perturbed star were to remain at the origin. In reality, the star's center star will not be at the coordinate origin: the orbiting body will move it to some point in the orbital plane. However, the magnitude of this shift must be of the same order as the perturbation itself: $r_{\rm cent} = \epsilon\xi(t)$ for some function $\xi(t)$. Now evaluate the density at the star's center with a Taylor expansion:

$$\begin{aligned} n_{\rm cent} &= n^0(0) + \epsilon\xi \left. \frac{dn^0}{dr} \right|_{r=0} \\ &+ \epsilon \sum_{l,m,\omega} \left[\delta n_{lm\omega}(0) + \epsilon\xi \left. \frac{d\delta n_{lm\omega}}{dr} \right|_{r=0} \right] Y_{lm}(\theta,\phi) e^{i\omega t} \; . \quad (7) \end{aligned}$$

As we have already shown, $\delta n_{lm\omega}(0) = 0$, except possibly for $l = 0$, while $dn^0/dr \to 0$ as $r \to 0$ by Eq. (4). Thus, the baryon density at the center of mass is given by

$$n_{\rm cent} = n^0(0) + \epsilon \delta n_{000}(0) + O(\epsilon^2) \; . \quad (8)$$

Only the monopole can produce changes in the central density which scale linearly in μ/R.

It is straightforward to solve for the $l = m = 0$ corrections to the metricoutside the fluid. Define the function

$$H(r) = \begin{cases} 0 & r < R \\ \dfrac{2\mu}{r} \dfrac{(1 - 2M/R)}{(1 - 3M/R)^{1/2}} & r > R \end{cases} \; ; \quad (9)$$

then $h_{tt} = H(r)$, $h_{rr} = H(r)/(1 - 2M/r)^2$. We have set $h_{\theta\theta} = h_{\phi\phi} = 0$ using the first order gauge freedom available for the monopole. As the point particle spirals into the star, some of its total mass-energy (its contribution to the total mass as measured at infinity) is radiated away, though its locally measured mass (rest mass) is conserved. The multiplicative factor in the above expression correctly accounts for that radiation loss. Notice that there is no monopole contribution to the metric inside the orbital radius of the particle—Keplerian orbits inside the orbital radius measure *only* the mass of the unperturbed neutron star at monopole order.

Is it possible for the monopole part of the perturbation to rearrange the fluid in the star, but leave its total gravitational mass unchanged? The answer is unequivocally no. A monopole perturbation is spherically symmetric and can only take one spherical solution into another. However, when the equation of state is fixed, all spherical solutions are parameterized by the gravitational mass—for each value of the gravitational mass M there exists a unique spherical configuration of the star. Spherical solutions therefore exhibit a one-to-one correspondence between the gravitational mass and the central density of the star. *Since the gravitational mass M is unchanged at* monopole order, the central density cannot be affected either. Hence, the central energy density of a neutron star is unaffected by a binary companion at order μ/R. *There is no crushing effect which scales linearly with μ/R.*

Interestingly, it is easily shown that incorrectly imposing boundary conditions can lead to an increase in central density at order μ/R. If the total gravitational mass of the star *and particle* is held fixed in a sequence of quasi-equilibrium solutions (ignoring the gravitational radiation that causes the orbital radius to shrink), *and* the particle's locally measured mass (rest mass) is held fixed, then obviously the star's total mass and baryon mass must go up by an amount of order $\mu M/R$, contrary to how a real binary would behave. This mass increase will drive the central density up by a fractional amount of order μ/R, which is what the WMM simulations show. We have no evidence that this is what actually happens in the WMM simulations; it merely illustrates one way in which the observed density increase could arise.

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