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ABSTRACT

We estimate the signal-to-noise ratios (SNRs) that one would expect to measure from coalescing binary black hole (BBH) systems for the following three broadband gravitational-wave observatories: initial and advanced ground-based interferometers (LIGO/VIRGO) and space-based interferometers (LISA). We focus particularly on the highly relativistic and nonlinear *merger* portion of the gravitational-wave signal, which comes after the adiabatic *inspiral* portion and before the *ringdown* portion due to the quasinormal ringing of the final Kerr black hole.

Ground-based interferometers can do moderate SNR (a few tens), moderate accuracy studies of the dynamics of merging black holes in the mass range (a few)$M_\odot$ to $\sim 2000 M_\odot$. LISA, by contrast, can do high SNR (a few $\times 10^4$), high-accuracy studies of BBH systems in the mass range $10^5 M_\odot \lesssim (1+z)M \lesssim 10^8 M_\odot$, where z is the binaries' cosmological redshift.

Our estimated SNRs suggest that coalescing black holes might well be the first sources detected by the LIGO/VIRGO network of ground-based interferometers. Because of their larger masses, they can be seen out to much greater distances (up to $\sim 500\,$Mpc for the initial LIGO interferometers) than coalescing neutron star binaries (heretofore regarded as the "bread and butter" workhorse source for LIGO/VIRGO, visible to $\sim 30\,$Mpc by the initial LIGO interferometers).

Low-mass BBHs ($M \lesssim 50 M_\odot$ for the first LIGO interferometers; $M \lesssim 100 M_\odot$ for the advanced; $(1+z)M \lesssim 3 \times 10^6 M_\odot$ for LISA) are best searched for via their well-understood inspiral waves; more massive BBHs must be searched for via their more poorly understood merger waves and/or their well-understood ringdown waves. A search for low-mass BBHs based on the inspiral waves, at a sensitivity level roughly half way between the first LIGO interferometers and the advanced LIGO interferometers, should be capable of finding BBHs out to $\sim 1\,$Gpc. A search for massive BBHs based on the ringdown waves can be performed using the method of matched filters. If one wants the reduction in the event rate due to the discreteness of the template family to be no more than 10%, then the number of independent templates needed in the search will only be about 6000 or less. Such a search with the first LIGO interferometers should be capable of finding BBHs in the mass range from about $100 M_\odot$ to about $700 M_\odot$ out to $\sim 200\,$Mpc; with advanced LIGO interferometers, from about $200 M_\odot$ to about $3000 M_\odot$ out to $z \sim 1$; and with LISA, BBHs with $10^6 M_\odot \lesssim (1+z)M \lesssim 3 \times 10^8 M_\odot$ should be visible out to $z \gtrsim 100$. The effectiveness of a search based on the merger waves will depend on how much one has learned about the merger waveforms from numerical relativity simulations. With only a knowledge of the merger waves' range of frequency bands and range of temporal durations, a search based on merger waves can be performed using a nonlinear filtering search algorithm. Such a search should increase the number of discovered BBHs by a factor of roughly 10 over those found from the inspiral and ringdown waves. On the other hand, a full set of merger templates based on numerical relativity simulations could further increase the number of discovered BBHs by a additional factor of up to about 4.

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I. INTRODUCTION AND SUMMARY

A. Coalescences of black hole binaries

It has long been recognized that coalescences of binary systems of two black holes could be an important source of gravitational waves [1,2], both for the ground based interferometric detectors LIGO [3] and VIRGO [4] currently under construction, and also for the possible future space-based interferometer LISA [5–7]. The orbits of binary black holes (BBHs) gradually decay from energy and angular momentum loss to gravitational radiation. Eventually, the holes coalesce to form a final black hole. For gravitational radiation reaction to successfully drive the binary to merge in less than a Hubble time, the initial orbital period must be $\lesssim 0.3\,\mathrm{days}\,(M/M_\odot)^{5/8}$, where M is the total mass of the binary; thus the critical orbital period is of the order of days for solar mass black holes, and of the order of years to hundreds of years for supermassive black holes ($10^6 M_\odot \lesssim M \lesssim 10^9 M_\odot$).

The process of coalescence can be divided up into three more or less distinct phases:

- An adiabatic *inspiral*, during which the gravitational radiation reaction timescale is much longer than the orbital period. The inspiral ends when the binary orbit becomes relativistically dynamically unstable at an orbital separation of $r \sim 6M$ (in units where $G = c = 1$) [8,9]. The gravitational waves from the inspiral carry encoded within them the masses and spins of the two black holes, some of the orbital elements of the binary and the distance to the binary [1,10].

- Towards the end of the inspiral, the black holes encounter the dynamical instability and make a gradual transition from a radiation-reaction driven inspiral to a freely-falling plunge [8,11,12], after which, even if the radiation reaction could be turned off, the holes would still merge. We will call the subsequent plunge and violent collision the *merger* phase. Gravitational waves from the merger could be rich with information about the dynamics of relativistic gravity in a highly nonlinear, highly dynamic regime which is poorly understood theoretically today.

- As the final black hole settles down to a stationary Kerr state, the nonlinear dynamics of the merger gradually become more and more describable as oscillations of this final black hole's quasinormal modes [13,14]. The corresponding emitted gravitational waves consist of a superposition of exponentially damped sinusoids. We will call the phase of the coalescence for which the emitted gravitational-wave signal is dominated by the strongest $l = m = 2$ quasinormal mode signal the *ringdown* phase. The waves from the ringdown carry information about the mass and spin of the final black hole [15,16]. (Note that, for want of a better terminology, throughout this paper we consistently use *coalescence* to refer to the entire process of inspiral, merger and ringdown, and reserve the word merger for the phase intermediate between inspiral and ringdown.)

In this paper we will for simplicity restrict attention to black hole binaries of approximately equal masses. We will consider three different classes of BBHs:

(i) *Solar mass* black hole binaries, *i.e.*, binaries that are formed from massive main-sequence progenitor binary stellar systems. Such BBHs are expected to have total masses in the range $10M_\odot \lesssim M \lesssim 50M_\odot$, but not much larger than this. The rate of coalescence of solar-mass BBHs in the Universe is not very well known. However, theory suggests that most of the BBH progenitor systems may not disrupt during the stellar collapses that produce the black holes, so that their coalescence rate could be about the same as the birth rate for their progenitors, about 1/100,000 years in our Galaxy, or several per year within a distance of 200 Mpc [17–21]. Note that this coalescence rate is roughly the same as the expected event rate for what has traditionally been regarded as the most promising source for ground based interferometers, coalescences of neutron star–neutron star (NS-NS) binaries [3,10]. However, the expected rate of NS-NS coalescences is much more firmly known, as it is based on extrapolations from detected progenitor NS-NS systems [17–19].

(ii) *Intermediate mass* black hole binaries, with total masses in the range $50M_\odot \lesssim M \lesssim (\text{a few}) \times 10^3 M_\odot$. In contrast to the cases of solar mass black holes and supermassive black holes (discussed below), there is little direct observational evidence for the existence of black holes in this mass range. Although there have been suggestions that the globular cluster M15 harbors a black hole of mass $\sim 10^3 M_\odot$ [22], theoretical modeling combined with the most recent HST observations neither confirm nor rule out this possibility [23]. Despite the lack of observational evidence, it is plausible that black holes in this mass range are formed in the cores of globular clusters, or in galactic nuclei in the process of formation of a supermassive black hole [26]. Simulations by Quinlan and Shapiro suggest that black holes with $M \sim 100M_\odot - 1000M_\odot$ could be formed in the evolution of dense stellar clusters of main sequence stars in galactic nuclei [24], and coalescences of binaries of such black holes could be possible en route to the formation of a supermassive black hole.

In this paper, we are prompted to consider BBHs in this intermediate mass range because of the following fact (see Sec. I E below): even if the coalescence rate of intermediate mass BBHs is $\sim 10^{-4}$ that of NS-NS binaries (which is thought to be $\sim 10^{-5}\,\mathrm{yr}^{-1}$ in our Galaxy as discussed above), these sources would still be seen more often than NS-NS binaries by LIGO's initial and advanced interferometers, and thus could be the first detected type of source.

(iii) *Supermassive* black holes binaries: There is a variety of strong circumstantial evidence that supermassive black holes (SMBHs) in the mass range $10^6 - 10^9 M_\odot$ are present in quasars and active galactic nuclei, and also that $\sim 25\% - 50\%$ of nearby massive spiral and elliptical galaxies harbor quiescent SMBHs [25,7]. See Ref. [26] for a review of this evidence. One of the main scientific goals of the LISA project is to detect and monitor various processes involving SMBHs, such as their formation [2,7], and the capture of compact stars by SMBHs [2,7,10,27,28]. In particular, the coalescences of SMBH binaries that are formed in Galaxy mergers, in which the individual SMBHs are driven together by dynamical friction and gas accretion until gravitational radiation reaction takes over [29], have often been suggested as a promising source for space-based interferometers [1,2,7,10,30,31]. Such coalescences would be detectable throughout the observable universe with large signal to noise ratios [7,10]. There is also observational evidence for SMBH binaries: wiggles in the radio jet of QSO 1928+738 have been attributed to the orbital motion of a SMBH binary [32], as have time variations in quasar luminosities [33] and in emission line redshifts [34]. The overall event rate is uncertain, but could be large ($\gtrsim 1/\text{yr}$), especially if the hierarchical scenario for structure formation is correct [31].

B. Status of theoretical calculations of the gravitational-wave signal

Detailed theoretical understanding and predictions of the gravitational waveforms $h_+(t)$ and $h_\times(t)$ produced in BBH coalescences will facilitate both the detection of the gravitational-wave signal, and the extraction of its useful information. In situations where a complete family of theoretical template waveforms is available, it will be possible to use the procedure of Wiener optimal filtering to search the interferometer data streams and to detect the gravitational-wave signal [1,35]. The resulting signal-to-noise ratios (SNRs) can be larger than those obtainable without theoretical templates by a substantial factor; see Sec. II below. Thus, while it is possible to detect the various phases of BBH coalescences without theoretical templates, such templates can greatly increase the effective range of the interferometers and the event detection rate. (Accurate theoretical templates are also essential for extracting the maximum amount of information from detected signals [36].)

Such theoretical template waveforms are available for the inspiral and ringdown phases of the coalescence, but not yet for the merger phase, as we now discuss.

For the inspiral phase of the coalescence, the gravitational waves and orbital evolution can be described reasonably well using the post-Newtonian approximation to general relativity. To date, inspiral waveforms have been calculated up to post-2.5-Newtonian order [37], and the prospects look good for obtaining waveforms up to post-3.5-Newtonian order [38,39]. Post-Newtonian templates will be fairly accurate over most of the inspiral, the most important error being a cumulative phase lag [40,41]. This cumulative phase lag will not be important for searches for inspiral waves; template phasing error will be largely compensated for by systematic errors in best-fit values of the binary's parameters, and the signals will still be found [40,42–44]. However, the template inaccuracies will be significant when one attempts to extract from the data the binary's parameters. In particular, post-Newtonian templates' errors start to become very significant around an orbital separation of $r \sim 12M$ [45], well before the end of the inspiral at the dynamical orbital instability ($r \sim 6M$). Templates for the phase of the inspiral between roughly $12M$ and $6M$ will most likely have to be calculated using methods other than the post-Newtonian approximation. Moreover, the methods of full blown numerical relativity cannot be applied to this "Intermediate Binary Black Hole" (IBBH) phase, since the total time taken to evolve from $12M$ to $6M$ is about $1500M$, too long for supercomputer simulations to even contemplate evolving. Alternative analytic and numerical methods for calculating gravitational waveforms from the IBBH portion of BBH inspirals, based on the adiabatic approximation, are under development [46]; it is likely that such alternative methods will be successfully developed and implemented before the gravitational-wave detectors begin their measurements [47].

Waveforms from the dynamic, complicated merger phase of the coalescence can only be obtained from numerical relativity. Unlike mergers of neutron star binaries, BBH mergers are particularly clean in the sense that there is no microphysics or hydrodynamics to complicate simulations of the evolution, and external perturbations are negligible: the entire merger can be described as a solution to the vacuum Einstein equation [49]. However, finding that solution is not a particularly easy task: a major computational effort to evolve the vacuum Einstein equation for BBH mergers using massive computational resources is currently underway, funded under the auspices of the Nation Science Foundation's Grand Challenge program [50,51].

The ringdown phase of the coalescence can be accurately described using perturbation theory on the Kerr spacetime background [52]. The gravitational waveforms from this phase are well understood, being just exponentially damped sinusoids. Thus, Wiener optimal filtering is feasible for searches for ringdown waves.

C. Purpose of this paper

The principal purpose of this paper is to estimate, in more detail than has been done previously, the prospects for measuring gravitational waves from the three different phases of coalescence events (inspiral, merger and ringdown), for various different detectors, and for a wide range of BBH masses. We estimate in each case the distances to which the different types of source can be seen by calculating expected SNRs. In particular, we determine for each BBH mass and each detector whether a coalescence event is most effectively detected by searching for the inspiral portion of the signal, or the merger portion, or the ringdown portion. We also determine how much the availability of theoretical template waveforms for the merger phase could increase the event detection rate.

Previous estimates of SNRs for ground-based interferometers have focused on the inspiral [1,40] and ringdown [15,16] phases, and also focused on solar-mass BBHs. For space-based interferometers, previous estimates of SNRs from the merger phase [7,10] were restricted to specific mass values and did not consider the ringdown portion of the signal.

In a companion paper, paper II of this series, we discuss in detail the useful information carried by the three phases of the gravitational-wave signal, and methods and prospects for extracting this information both with and without templates for the merger phase [36].

In the following subsection we describe our calculations and summarize the assumptions underlying our estimated signal-to-noise ratios. In Sec. I E below we summarize our main results and conclusions, and an outline of the paper is given in Sec. I F.

D. Estimating the signal-to-noise ratios: method and assumptions

We calculate SNRs for three different types of interferometer: initial and advanced ground-based interferometers (LIGO/VIRGO), and the proposed space-based interferometer LISA. The noise spectra of the initial and advanced ground-based interferometers we took from Ref. [3], and that for LISA we obtained from Ref. [7]. Our approximate versions of these noise spectra are given in Eqs. (4.1) - (4.4) below, and are illustrated in Figs. 1 - 3 in Sec. V A.

We consider the following three different signal-detection methods:

(i) *Matched filtering searches:* For those phases of the coalescence for which a complete set of theoretical templates will be available (*i.e.*, the inspiral, the ringdown, and possibly the merger), the method of matched filtering or optimal filtering can be used to search for the waves [1,35,53–55]. The theory of matched filtering is briefly sketched in Sec. II A below. For any source of waves, the SNR, ρ, obtained from matched filtering is related to the gravitational waveform $h(t)$ measured by the interferometer and to the spectral density $S_h(f)$ of the strain noise in the interferometer via [56]

$$\rho^2 = 4\int_0^\infty \frac{|\tilde{h}(f)|^2}{S_h(f)} df, \tag{1.1}$$

where $\tilde{h}(f)$ is the Fourier transform of $h(t)$ defined by Eq. (2.3) below. The SNR (1.1) depends, through the waveform $h(t)$, on the orientation and position of the source relative to the interferometer. In Sec. II C below we show that if we perform an rms average over source orientations and positions (at a fixed distance), the rms SNR thus obtained depends only on the energy spectrum dE/df carried off from the source by the gravitational waves. The resulting relationship between the waves' energy spectrum and the rms angle-averaged SNR forms the basis for most of our calculations. It is given by [*cf.* Eq. (2.34) below]

$$\langle\rho^2\rangle = \frac{2(1+z)^2}{5\pi^2 D(z)^2}\int_0^\infty df\,\frac{1}{f^2 S_h(f)}\,\frac{dE}{df}[(1+z)f], \tag{1.2}$$

where z is the source's cosmological redshift and $D(z)$ is the luminosity distance from the source.

In order for a signal to be detected, the waves' measured SNR must be larger than the detection threshold

$$\rho_{\rm threshold} \approx \sqrt{2\ln(\mathcal{N}_{\rm start-times}) + 2\ln(\mathcal{N}_{\rm shapes})}; \tag{1.3}$$

see, for example, Ref. [40] and also Sec. II B below. Here $\mathcal{N}_{\rm start-times}$ is the number of independent starting times of the gravitational wave signal that are searched for in the data set, determined by the total duration of the data set (of order one year) and the sampling time. The quantity $\mathcal{N}_{\rm shapes}$ is the total number of trial waveform shapes used in the matched filtering process.

(ii) *Band-pass filtering searches:* For the merger phase, a complete set of theoretical templates may not be available, in which case it will not be possible to use matched filtering and other search methods will need to be employed. Band-pass filtering, followed by setting a detection threshold in the time domain, is a simple method of searching an interferometer data stream for bursts of unknown form [35]. In Sec. II A below we derive an approximate relation between the SNR obtainable from bandpass filtering, and the SNR (1.1) obtainable from matched filtering, for any burst of waves, namely

$$\left(\frac{S}{N}\right)_{\rm band-pass} \approx \frac{1}{\sqrt{2T\Delta f}}\left(\frac{S}{N}\right)_{\rm optimal}. \tag{1.4}$$

Here T is the duration of the burst and Δf is the bandwidth of the bandpass filter [*cf.* Eq. (2.14) below]. The quantity $2T\Delta f$ is the dimension of the linear space of signals being searched for, which is roughly the same as the "number of cycles" of the gravitational waveform. We use this formula in Sec. VI B below to estimate bandpass-filter search SNRs for the merger waves from BBH coalescences, by inserting on the right hand side the rms angle-averaged matched-filter search SNR (1.2), and by making estimates of T and Δf.

(iii) *Nonlinear filtering searches:* The traditional view has been that the SNR (1.4) is about the best that can be achieved in the absence of theoretical templates, that is, that the gain in SNR obtainable from matched filtering is approximately the square root of the number of cycles in the gravitational wave signal. This view is based on the the assumption that the search method used in the absence of templates is bandpass filtering or something very *similar*. However, we suggest in Sec. II B below an alternative search method, motivated by Bayesian analyses and based on nonlinear filtering, which performs much better than bandpass filtering and in some cases almost as well as matched filtering. This search method is also described in detail in Ref. [57]. It is based on calculating the quantity

$$Q(t) = \frac{1}{T}\int_{-T/2}^{T/2} d\tau\, h_1(t+\tau)^2, \tag{1.5}$$

where T is the maximum expected duration of the signal, and $h_1(t)$ is a suitably prefiltered version of the interferometer data stream.

The efficiency of this nonlinear filtering method cannot usefully be described in terms of a signal-to-noise ratio, since the detection statistic is very non-Gaussian. Instead, its efficiency can be described in the following way. Let ρ denote the SNR that would be obtained by matched filtering [Eq. (1.1) above], which we use as a convenient parameterization of the signal strength, and which is meaningful even in situations where templates are not available and where optimal matched cannot be carried out. Then, a signal will be detected with high confidence using nonlinear filtering whenever ρ is larger than the threshold ρ_*, where to a good approximation ρ_* satisfies

$$\rho_*^2 = 2\ln(\mathcal{N}_{\rm start-times}) + \mathcal{N}_{\rm bins}\ln\left(1+\rho_*^2/\mathcal{N}_{\rm bins}\right). \tag{1.6}$$

Here $\mathcal{N}_{\rm bins} = 2T\Delta f$ is the dimension of the signal space discussed above (or the number of independent frequency bins in the Fourier domain). The derivation of Eq. (1.6) is given in Sec. II B below, together with a simple, intuitive way of understanding it.

The relation (1.2) forms the basis of our SNR calculations, and the SNR thresholds (1.3) and (1.6) underlie our deductions from the calculated SNR values as to the detectability of the various types of gravitational-wave signal. To calculate the SNRs, we also need to specify the waves' energy spectra for the three different phases of the coalescence. As we now outline, for the inspiral and ringdown phases the waves' energy spectrum is essentially known, while for the merger phase we make an educated guess of dE/df. Sec. III below gives more details.

Inspiral energy spectrum: We use the leading order expression for dE/df obtained using Newtonian gravity supplemented with the quadrupole formula [58] [Eq. (3.19) below]. Strictly speaking, this spectrum describes the SNR that would be achieved by searching for Newtonian, quadrupole waves using Newtonian, quadrupole templates. However, the actual SNR obtained when searching for a real, general-relativistic inspiral waveform using post-Newtonian templates should deviate from this by only a few tens of percent [59]. We terminate the spectrum at the frequency $f_{\rm merge} = 0.02/M$ which is (roughly) the frequency of quadrupole waves emitted at the orbital dynamical instability at $r \sim 6M$ [8]. For LISA, we assume that the measurement process lasts at most one year, and choose the frequency at which the inspiral spectrum starts accordingly.

Ringdown energy spectrum: The spectrum that we use [Eq. (3.23) below] is determined (up to its overall amplitude) by the characteristics of the $l = m = 2$ quasi-normal ringing (QNR) mode of the final Kerr black hole. This mode is the most slowly damped of all QNR modes, so we expect it to dominate the last stages of the gravitational-wave emission. The QNR spectrum depends on three parameters: the quasi-normal modes' frequency $f_{\rm qnr}$ and damping time τ, and the overall amplitude of the quasinormal mode signal. Equivalently, the three parameters can be taken to be the mass M and dimensionless spin parameter a of the final black hole (which determine $f_{\rm qnr}$ and τ) and the total amount of energy radiated in the ringdown (which determines the overall amplitude). The spectrum is peaked at $f = f_{\rm qnr}$ with width $\Delta f \sim 1/\tau$.

In our analyses, we (somewhat arbitrarily) assume that $a = 0.98$. It seems likely that in many coalescences the spin of the final black hole will be close to maximal, since the total angular momentum of the binary at the end of the inspiral is $\sim 0.9M^2$ when the individual black holes are non-spinning [61], and the individual black hole spins can add to this. Exactly how close to maximally spinning the final black hole will be is a matter that probably will not be decided until supercomputer simulations—or observations—settle the issue. In any case, the ringdown SNR *values that we obtain depend only weakly* on our assumed value of a [*cf.* Eq. (A14) below].

The overall amplitude of the ringdown signal depends upon one's delineation of where "merger" ends and "ringdown" begins, which is somewhat arbitrary. We assume a value of the overall amplitude that corresponds to a total radiated energy in the ringdown of $0.03M$, *i.e.*, a 3% radiation efficiency. This number is based on a back of the envelope, quadrupole-formula-based estimate of the QNR mode's amplitude when the distortion of the horizon of the black hole is of order unity (*cf.* Sec. III D below). Although this radiation efficiency seems rather high, there have been numerical simulations of the evolution of dis-