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The Stochastic Gravity-Wave Background: Sources and Detection

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1. INTRODUCTION

The design and construction of a number of new and more sensitive detectors of gravitational radiation is currently underway. These include the LIGO detector being built in the United States by a joint Caltech/MIT collaboration [1], the VIRGO detector being built near Pisa by an Italian/French collaboration [2], the GEO-600 detector being built in Hannover by an Anglo/German collaboration [3], and the TAMA-300 detector being built near Tokyo [4]. There are also several resonant bar detectors currently in operation, and several more refined bar and interferometric detectors presently in the planning and proposal stages.

It's not clear when the first sources will be detected – this may happen soon after the detectors go "on-line" or it may require decades of further work to increase the sensitivity of the instruments. But it's clear to me that eventually, when the sensitivity passes above some threshold value, the gravity-wave detectors will find sources.

The operation of these detectors will have a major impact on the field of gravitational physics. For the first time, there will be a significant amount of experimental data to be analyzed, and the "ivory tower" relativists will be forced to interact with a broad range of experimenters and data analysts to extract the interesting physics from the data stream. Being an optimist, I think that this "interesting physics" will include sources of gravitational radiation which we have not hitherto expected or even conceived of. It promises to be an exciting time.

There are at least three major types of "known" sources [5, 6]. These are (1) coalescing binary systems, composed of neutron stars and/or black holes, (2) pulsars (and other periodic sources), and (3) supernovae (and other transient or burst sources). These are listed, roughly speaking, in increasing order of "detection difficulty". For example, the waveforms (chirps) of the coalescing binary systems are already known to a high enough degree of precision [7, 8, 9] to enable reliable detection, using the classical signal analysis technique of matched filtering [10, 11]. Some order-of-magnitude studies have shown that the amount of required processing can be done with a dozen or so high-performance desktop workstations [12]. The major unknown for this type of source is their number density, which is uncertain by more than an order of magnitude. Hence the rate of events which occur close enough to us to be observable can not be estimated very precisely. The second type of source is pulsars. Once again, the form of the signal is known very precisely: it's just a sine wave in the solar system barycenter, a coordinate system at rest with respect to the Sun [13]. In this case however, the signal processing problem is far more difficult, because the orbital motion of the earth around the sun and the rotational motion of the earth around its axis modulates the pulsar frequency. This means one must separately analyze the signal for each of $\approx 10^{14}$ separate patches on the celestial sphere, each of which would have a distinctive pattern of frequency modulation. The required processing speeds are currently beyond the limits of even the most powerful computers. Of course, it may be just a question of waiting until a better search algorithm is developed, or faster computers become available! The third source which I've listed, supernovae, is perhaps the most difficult to detect. There are two reasons. First, we do not have precise predictions of the gravitational waveform such an event would produce – and it's hard to design a data analysis algorithm to look for something unknown. Furthermore, these explosions are expected to be rare events and because certain kinds of instrument *noise might* look quite similar to supernovae, I chose to list them last, and categorize them as the most difficult sources to detect.

The subject of these lectures is a fourth type of source, quite different in character from the three listed above. These are the "stochastic" or "background" sources [14, 15, 16]. Roughly speaking, these are "random" sources, typically arising from an extremely large number of "unresolved" independent and uncorrelated events. This type of background could be the result of processes that take place very shortly after the big bang, but since we know very little about the state of the universe at that time, it's impossible to say with any certainty. Such a background might also arise from processes that take place fairly recently (say within the past several billion years) and this more recent contribution might overwhelm the parts of the background which contain information about the state of the early universe.

These sources are "unresolved" in the following sense. If we study an optical source, somewhere in the sky, using a telescope with a certain angular resolution, then details of the source can be "resolved" if the angular resolution of the telescope is smaller than the angular size of the features or objects being studied. In the case of the LIGO experiment, and similar detectors, the angular size of the antenna pattern is of order 90°. Hence almost any source is "unresolved" in that it makes a significant contribution to the detector output for almost any orientation of the detector and the source. When many such sources are present, even if they are point-like, the resulting signal has a stochastic nature.

My motivation for studying these stochastic sources is two-fold. The first reason is a rather hopeful one. Because the gravitational force is the weakest of the four known forces, the small-scale perturbations of the gravitational field decouple from the evolution of the rest of the universe at very early times. Currently, our most detailed view of the early universe comes from the microwave background radiation, which decoupled from matter about 10^5 years after the big bang, and gives us an accurate picture of the universe at this rather early time. Some rather simple estimates (which I'll elaborate later) show that if the current crop of gravity-wave detectors do detect a background of cosmological origin, then it will carry with it a picture of the universe as it was about 10^{-22} seconds after the big bang. This would represent a tremendous step forward in our knowledge, and is the main reason for my interest.

The second reason for my interest in stochastic sources is rather more practical. One often hears it said that searching for signals in the output of a gravitational wave antenna is like searching for a needle in a haystack. And indeed, for the first three types of source listed above, this is true. One has to search through tons of rock (the data stream) in order to find the one precious gem (say, a binary chirp). Most of the rock is barren, and the challenge is to isolate the one tiny volume containing the material of interest. As I have already discussed, for gravitational wave sources, the amount of computational power required for this careful search can be very large. However the situation is rather different when one is searching for a stochastic background. The analogy in this case is mining aluminum, where on average every ton of ore contains a certain number of kilograms of aluminum. The situation is analogous for stochastic gravity-wave sources. As you will see, whenever two detectors are operating simultaneously, even if only for a few seconds, we get a little bit more data and information about the stochastic background. And as you will also see, it is easy to analyze this data. The "signal" in this case is a very low bandwidth one, so the essential part of the data analysis for a stochastic background can be done on a garden-variety personal computer. For example, in the case of the LIGO detectors, the part of the signal carrying a significant amount of information lies below a few hundred Hz (see Section 3.3 and Fig. 7 for details). Thus the rate at which information needs to be processed is only a few hundred data points/second; a very manageable rate.

These lectures are organized as follows. In Section 2, I discuss some of the general properties that a stochastic background of gravitational radiation might have. I show how such radiation is characterized by a spectral function, discuss some of its statistical properties, and show during which cosmological epoch the radiation which falls into the bandwidth of the ground- and space-based detectors was produced. In Section 3 I show how one can combine data from two or more gravity-wave detectors to either put limits on the amplitude of a stochastic background, or to actually detect it. I do this in several steps, first giving a crude argument which demonstrates the main detection strategy, then discussing the reduction in sensitivity which comes about from the siting of the detectors and their relative orientations on the earth. This is followed by a rigorous derivation of the optimal signal processing strategy, and a calculation of the expected signal-to-noise ratio and the minimum detectable energy-density in a stochastic background. In Section 4 I discuss the observational facts: what we actually know about the stochastic background. There are strong limits on the spectrum of a stochastic background coming at very long wavelengths from observations of the isotropy of the Cosmic Microwave Background Radiation, and limits at higher frequencies from observations of timing residuals in millisecond pulsars. Finally, there is a limit on the integrated spectrum arising from the standard model of big bang nucleosynthesis. The last part of these lectures is far more speculative. In Section 5 I discuss some of the potential ways in which a stochastic background might arise from processes that take place early in the history of the universe. I discuss in some detail three particular models (inflation, cosmic strings, and bubble formation in a first-order phase transition) each of which gives rise to a different spectrum of radiation, and examine in some detail the mechanisms at work in each case. In each case I indicate if and when the experiments being built might detect these potential sources. This is followed by a short conclusion, and an appendix containing a few calculational details.

2. THE STOCHASTIC BACKGROUND: SPECTRUM & PROPERTIES

2.1. Digression - the Cosmic Microwave Background Radiation

In order to discuss the spectrum of gravitational background radiation, we need to introduce notation. I will do this with an analogy and an example.

Let's begin by considering the electromagnetic background radiation, conventionally referred to as the Cosmic Microwave Background Radiation (CMBR) [17]. This radiation was originally produced when the universe had a temperature of about 3000 K, at a red-shift of approximately $Z = 1100$. Today, this electromagnetic radiation has a Planck blackbody spectrum with a temperature of about $T = 2.73\,\mathrm{K}$. Each cubic centimeter around us contains ≈ 400 CMBR photons of energy $\approx 10^{-15}$ erg. The total energy density in this radiation field is $\rho_{\rm em} = 4.2 \times 10^{-13}\,\mathrm{ergs/cm^3}$, and the characteristic frequency of the photons that make it up is (a few times) $f_0 = kT/h = 5.7 \times 10^{10}\,\mathrm{Hz}$. Here

$h = 6.6 \times 10^{-27}$ erg − sec is Planck's constant, $k = 1.4 \times 10^{-16}$ erg/Kelvin is Boltzmann's constant.

In order to characterize the spectral properties of this radiation, we'll now consider how this energy is distributed in frequency. In a spatial volume V, the amount of energy dE contained in this radiation field between frequencies f and $f + df$ can be expressed as

$$dE = (2)hf\left(\frac{1}{e^{hf/kT}-1}\right)\left(\frac{4\pi V f^2 df}{c^3}\right), \tag{1}$$

where $c = 3 \times 10^{10}$ cm/sec is the speed of light. The different factors appearing on the right hand side of this equation are (1) the number of polarizations (2) the energy per quanta (3) the number of quanta per mode, and (4) the number of modes in the frequency interval. Dividing both sides of this equation by the volume V, one may write the energy density within the frequency range df as

$$d\rho_{\rm em} \equiv \frac{dE}{V} = \frac{8\pi h}{c^3}\frac{f^3 df}{e^{hf/kT}-1}. \tag{2}$$

It will prove convenient to write the differential energy density within a unit *logarithmic* frequency interval. This is

$$\frac{d\rho_{\rm em}}{d\ln f} = f\frac{d\rho_{\rm em}}{df} \approx 3.8 \times 10^{-14}\frac{\rm ergs}{\rm cm^3}\left(\frac{f}{f_0}\right)^4\left(\frac{e-1}{e^{f/f_0}-1}\right). \tag{3}$$

This formula contains complete information about the spectral distribution of energy in the CMBR, and if we were interested in discussing electromagnetic backgrounds, we would probably stop here. However there is a slightly different standard convention used to describe gravitational wave backgrounds.

In describing gravitational wave stochastic backgrounds, it is conventional to *compare* the energy density to the critical energy density $\rho_{\rm critical}$ required (today) to close the universe. This critical energy density is determined by the rate at which the universe is expanding today. Let us denote the Hubble expansion rate today by

$$H_0 = h_{100}\, 100\, \frac{\rm Km}{\rm sec - Mpc} = 3.2 \times 10^{-18} h_{100}\frac{1}{\rm sec} = 1.1 \times 10^{-28} c h_{100}\frac{1}{\rm cm}. \tag{4}$$

Because we don't know an accurate value for H_0 (a matter of considerable controversy in the literature) we include a dimensionless factor of h_{100} which almost certainly lies within the range $1/2 < h_{100} < 1$. The critical energy-density required to just close the universe is then given by

$$\rho_{\rm critical} = \frac{3c^2 H_0^2}{8\pi G} \approx 1.6 \times 10^{-8} h_{100}^2 \,{\rm ergs/cm^3}. \tag{5}$$

This leads to our fundamental definition in this section, of a quantity which we will be using for the remainder of these lectures.

In the remainder of these lectures, we will be discussing gravitational wave, rather than electromagnetic backgrounds. However, to complete this section, we first define the analogous quantity for electromagnetic backgrounds. This quantity is a dimensionless function of frequency

$$\Omega_{\rm em}(f) \equiv \frac{1}{\rho_{\rm critical}} \frac{d\rho_{\rm em}}{d\ln f}. \tag{6}$$

For the thermal spectrum of electromagnetic radiation in the CMBR, we have

$$\Omega_{\rm em}(f) = 2.4 \times 10^{-6} h_{100}^{-2} \left(\frac{f}{f_0}\right)^4 \left(\frac{e-1}{e^{f/f_0}-1}\right). \tag{7}$$

A graph this function is shown in Fig. 1. Also shown in Fig. 1 is the spectrum that the gravitational wave stochastic background would have, if at early times in the history of the universe the fluctuations in the gravitational field had been in equilibrium with the other matter and radiation in the universe. In this case, the gravitational wave stochastic background would have a thermal spectrum, with a temperature of about 0.9 K [17]. It is smaller than the temperature of the CMBR because in a conventional hot big bang model, the gravitons would have decoupled when the temperature of the universe dropped below the Planck temperature, when the number of entropy degrees of freedom was 106.75 in the standard GUT model. Since the number of degrees of freedom today is only 3.91, the graviton temperature is less than that of the CMBR by the ratio $(106.75/3.91)^{1/3}$. (See discussion in [17], pg 75, between (3.89) and (3.90)). However it is unlikely that this equilibrium could have been established; the time required to establish the equilibrium is longer than the characteristic expansion time (the Hubble time) of the universe because the gravitational interaction is so weak. While it is therefore unlikely that this 0.9 K thermal spectrum is present, it is nevertheless a useful benchmark for comparison.

A number of graphs similar to Fig. 1 will appear in these lectures, so a couple of comments are in order. First, the reader will notice that the horizontal axis encompasses an *enormous* range of frequencies. The lowest frequencies $f \approx H_0$ are those of waves which only oscillate a single time in the entire history of the universe, and whose period is a Hubble time H_0^{-1}! The highest frequencies shown are those of visible light. Second, these graphs make it easy to see the total amount of energy contributed by the radiation to the energy density of the universe. From the vertical axis, one can immediately see from the graph of $\Omega_{\rm em}(f)$ that the CMBR contains, in the vicinity of 10^{11} Hz, about 10^{-5} of the energy required to close the universe.

2.2. Notation - the Spectral Function $\Omega(f)$ for Gravitational Waves

In order to characterize the spectrum of a stochastic gravitational wave signal, we introduce a quantity for the graviton background which is analogous to $\Omega_{\rm em}$.

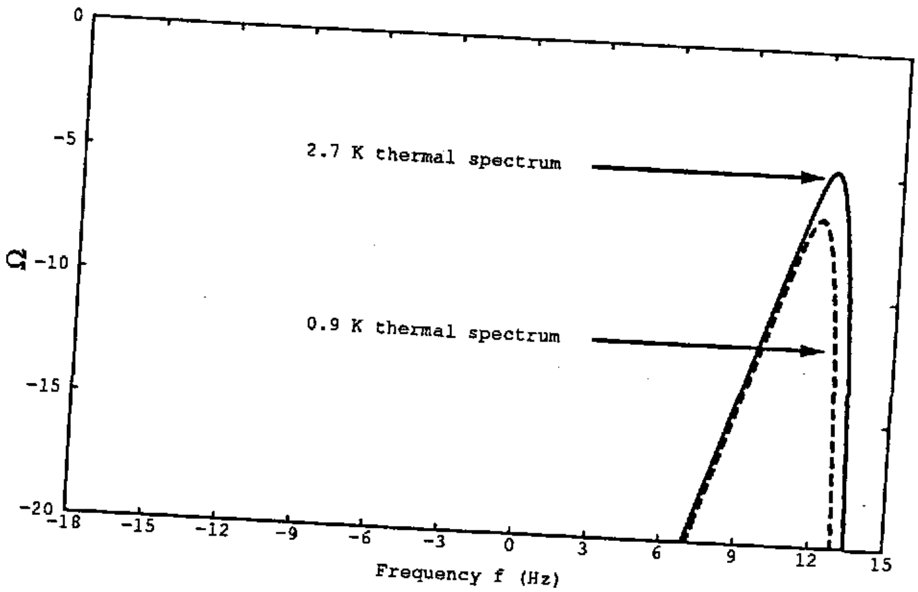

Fig. 1.— The solid curve is the fractional energy density $\Omega_{em}(f)$ contained in the 2.73 K electromagnetic background radiation (with h_{100} set to unity). The dashed curve shows the corresponding quantity for a 0.9 K blackbody. If the gravitational perturbations had been in equilibrium with the matter fields, this is the expected spectrum of the gravitational wave stochastic background. Both axes are $\log_{10}$.

The precise definition is

$$\Omega(f) = \Omega_{gw}(f) \equiv \frac{1}{\rho_{critical}} \frac{d\rho_{gw}}{d \ln f}. \tag{8}$$

The subscripts "gw" for "gravitational wave" are omitted when there is no danger of ambiguity.

There appears to be some confusion about $\Omega(f)$ in the literature. Some authors assume that $\Omega(f)$ is independent of frequency f; this is true for some cosmological models, but not for all of them. The important thing is that *any* spectrum of gravitational radiation can be described by an appropriate $\Omega(f)$. With the correct dependence on frequency f it can describe a flat spectrum, or a black-body spectrum, or any other specific distribution of energy with frequency.

You will also notice that it follows directly from the definition that the quantity $h_{100}^2 \Omega(f)$ is independent of the actual Hubble expansion rate. For this reason, we will often focus attention on that quantity, rather than on $\Omega(f)$ alone.

It is sometimes convenient to discuss the spectrum of gravitational waves and the sensitivity of detectors in terms of a characteristic "chirp" amplitude.

This is the dimensionless gravitational-wave strain $h = \Delta L/L$ that would be produced in the arms of a detector, in a bandwidth equal to the observation frequency. This is related to $\Omega(f)$ by

$$h_c(f) = 1.3 \times 10^{-20} h_{100} \sqrt{\Omega(f)} \frac{100\,\text{Hz}}{f}. \tag{9}$$

(Equation 65 of [18]). Apart from an overall factor, this formula can be easily derived by dropping the time derivatives from (A13). For example if $\Omega(f) = 10^{-8}$ over a bandwidth $50\,\text{Hz} < f < 150\,\text{Hz}$ then the strain in an ideal detector is $h_c \approx 10^{-24}$.

2.3. Assumptions about the Stochastic Gravitational-Wave Background

Does $\Omega(f)$ contain all information about the stochastic background? The answer is "yes" provided that we make enough additional assumptions.

We will assume from here on that the stochastic gravity wave background is isotropic, stationary, and Gaussian; under these conditions it is completely specified by its spectrum $\Omega(f)$. Each of these three properties might or might not hold; before moving on, let's consider each in turn.

It is now well established that the CMBR is highly isotropic [17]. In fact this isotropy is surprising, because in the standard model of cosmology, the angular size of the horizon at $Z = 1100$ is only about $2°$. Nevertheless, it is experimentally well-established that the largest deviation from the isotropy of the CMBR arises from our proper motion with respect to the rest frame of the universe, at the level of 1 part in 10^3. The next largest deviations from isotropy arise at the level of 1 part in 10^5; these fluctuations arise because of the non-uniform distribution of matter at (and after) the surface of last scattering.

It is therefore not unreasonable to assume that the stochastic gravity wave background is also isotropic. However this assumption may not be true. For example, suppose that the dominant source of stochastic gravity wave background is a large number of unresolved white dwarf binaries within our own galaxy. Because our galaxy is bar or spiral shaped (and not spherical) if we assume that the white dwarf binaries are distributed in space in the same way as the matter in the galaxy, then the stochastic background will have a distinctly anisotropic distribution, and will form a "band in the sky" distributed roughly in the same way as the milky way. It is also possible for a stochastic gravity wave background of cosmological origin to be quite anisotropic. Of course in this case, one is left with the problem of explaining why the CMBR is isotropic, but the gravity wave background is not. It is possible to conceive of such mechanisms, but within the scope of these lectures, we will not consider their effects.

The assumption that the stochastic background is stationary is almost certainly justified. Technically this means that the n-point correlation functions of the gravitational wave fields depend only upon the differences between the

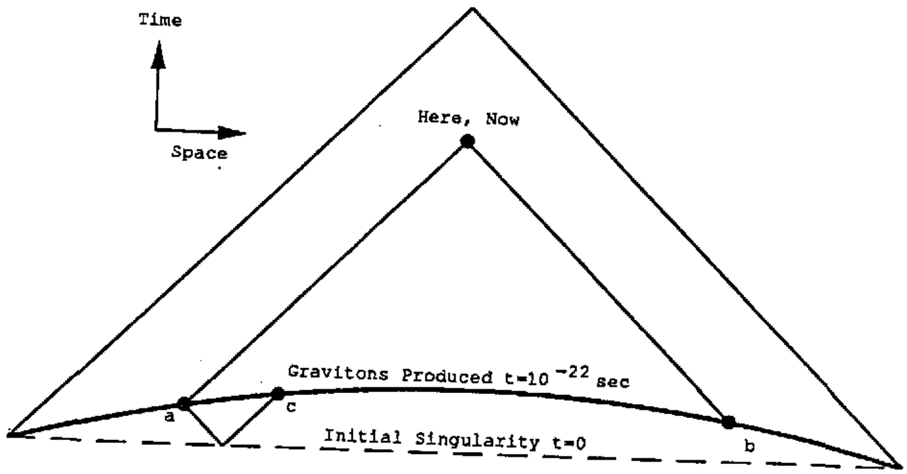

Fig. 2. — A conformal diagram showing a spatially-flat Friedman-Robertson-Walker cosmological model. The past light cone of a present-day observer (Here, Now) intersects a surface at $t = 10^{-22}$ seconds in a large 2-sphere, of which two points (a,b) are shown. The future light cone of the initial singularity intersects the same surface in a much smaller 2-sphere, of which two points (a,c) are also shown. The number of independent, uncorrelated horizon volumes $N_{\rm horizon}$ which contribute to the gravitational radiation arriving at a detector today is given by the ratio of the areas of the larger 2-sphere to the smaller one.

times, and not on the choice of the time origin. Because the age of the universe is 20 orders of magnitude larger than the characteristic period of the waves that LIGO, VIRGO, and the other facilities can detect, and 9 orders of magnitude larger than the longest realistic observation times, it seems very unlikely that the stochastic background has statistical properties that vary over either of these time-scales.

The final assumption we make is that the fields are Gaussian, by which we mean that the joint density function is a multivariate normal distribution. For many early-universe processes which give rise to a stochastic background, this is a reasonable assumption, which can be justified with the central limit theorem. Let's sketch the argument out.

We will show in Section 2.4 that if the stochastic background arises from processes that take place in the early universe, the characteristic time (proper time after the big bang) at which the gravitational radiation was emitted was about $t = 10^{-22}$ sec. Consider the conformal diagram [19], shown in Fig. 2. As Fig. 2 graphically illustrates, a detector today located at the spacetime point labeled "Here, Now" observes radiation produced at $t = 10^{-22}$ seconds after the big bang by an extremely large number $N_{\rm horizon}$ of independent horizon volumes. We can estimate the number of these horizon-sized volumes on the surface $t = 10^{-22}$ seconds as follows.

Let's assume that the universe is $k = 0$ (spatially flat) and radiation dominated from the time that the gravitons were produced ($t_1 = 10^{-22}$ sec) until the present time ($t_0 = 10^{17}$ sec). This is a reasonable assumption for this type of calculation; although the universe did recently become matter dominated (at a 1 of a few thousand) this has only a small effect on the final answer. The red-shift Z of the $t = t_1$ constant-time surface is then given by $1 + Z = (10^{17}/10^{-22})^{1/2} \approx 10^{20}$. Let's work in a "conformal time" coordinates where $ds^2 = a^2(\eta)(-d\eta^2 + d\vec{x}^2)$. For a radiation-dominated universe $a(\eta) = \eta$, and we can take the present time to have $\eta = \eta_0$ and the time at which gravitons were produced to have $\eta = \eta_1$. We begin by considering the intersection of the forward light cone of the initial singularity with the surface $\eta = \eta_1$. This intersection forms the small two-sphere denoted by the points "a" and "c" in Fig. 2; this two-sphere has area $A_{\rm small} = 4\pi\eta_1^2 a^2(\eta_1)$. In similar fashion, the past light cone of the present-day detector intersects the surface $\eta = \eta_1$ in the large two sphere denoted by the points "a" and "b", which has area $A_{\rm big} = 4\pi(\eta_0 - \eta_1)^2 a^2(\eta_1)$. The ratio of these two areas is the number of horizon volumes visible to the detector over the entire sky; it is

$$N_{\rm horizon} = \frac{A_{\rm big}}{A_{\rm small}} = \frac{(\eta_0 - \eta_1)^2}{\eta_1^2} = Z^2 \approx 10^{39}. \tag{10}$$

If we assume that the processes producing gravitational waves from each of these separate horizon volumes act independently, then it follows immediately from the central limit theorem that the amplitude of the radiation arriving at the detector, which is the sum of the amplitudes of the radiation produced by each of the separate volumes, is Gaussian.

2.4. When is a Stochastic Background produced?

Suppose that the LIGO or VIRGO detectors do detect a stochastic background of gravitational radiation. From what epoch, in the history of the universe, does this radiation date?

To give a general answer to this question, we need to make some assumptions about the universe. The most reasonable thing to do is to adopt the "standard model" of cosmology. This cosmological model consists of a spatially-flat ($k =$ 0 Friedman-Robertson-Walker) cosmological model, in which the equation of state is dominated by massless (or highly relativistic) matter for red-shifts greater than $Z_{\rm eq} \approx 6000$, and is dominated by massive pressureless particles (dust!) for smaller red-shifts.

The question posed above has two answers, one mundane, the other quite exciting. The mundane possibility is that the radiation was produced at or near the present epoch (say, within a red-shift $Z < 4$) by many unresolved separate sources, such as white dwarf binaries or supernovae. For example, in the case of the electromagnetic background radiation, one finds that below about 300 MHz the spectrum is dominated by (recent) emission from our own galaxy [20]. The