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Detection of Anisotropies in the Gravitational-Wave Stochastic Background

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By correlating the signals from a pair of gravitational-wave detectors, one can undertake sensitive searches for a stochastic background of gravitational radiation. If the stochastic background is anisotropic, then this correlated signal varies with the earth's rotation. We calculate how this varying signal is related to the amplitudes of the anisotropy multipole moments. The specific case of the two LIGO (Laser Interferometric Gravitational Observatory) detectors, which will begin operation around the year 2000, is analyzed in detail.

I. INTRODUCTION

The design and construction of a number of new and more sensitive detectors of gravitational radiation is currently underway. These include the LIGO detector being built in the United States by a joint Caltech/MIT collaboration [1], the VIRGO detector being built near Pisa by an Italian/French collaboration [2], the GEO-600 detector being built in Hannover by an Anglo/German collaboration [3], and the TAMA-300 detector being built near Tokyo [4]. There are also several resonant bar detectors currently in operation, and several more refined bar and interferometric detectors presently in the planning and proposal stages.

When two or more of these detectors are operating, it will become possible to correlate their signals, and in this way, to search for a stochastic background of gravitational radiation. The technique for such a search was originally described in work by Michelson [5], Cristensen [6] and Flanagan [7]. A review of these techniques may be found in [8]. Such radiation might be the result of processes that took place during the very early universe. It might also result from the incoherent superposition of many faint unresolvable present-day sources such as coalescing binary systems.

The stochastic gravitational-wave background might be isotropic on the sky, or it might be anisotropic. For example, if the background results from early-universe processes, then it might be isotropic to about the same degree as the 2.7° K electromagnetic background radiation. On the other hand, if the background is due to white-dwarf binaries in our own galaxy, then they might be distributed in a pancake or bar which mimics the shape of the observed luminous matter in our galaxy. In this paper, we show how the correlated signal from a pair of gravitational wave detectors is related to multipole moments which characterize the anisotropy. This should permit a signal to be analyzed to search for (or place upper limits on) the multipole moments which characterize the anisotropy. In this paper, we will assume that the reader is already familiar with the work previously cited (references [5–8]) on stochastic background detection.

These paper is organized as follows. In Section II we show how a background of stochastic gravitational radiation may be decomposed in a plane-wave expansion, with the coefficients of the expansion treated as stochastic random variables. In Section III the properties of these random variables are related to the (frequency) spectrum and spatial distribution of the radiation, and a set of multipole moments are introduced which characterize the anisotropies of the stochastic background. These anisotropies may be searched for by studying the variations of the detector outputs as the earth rotates relative to the fixed cosmic frame. In Section IV we show how the correlation between a pair of detectors fixed on the earth varies with time as the earth rotates, and detail how that correlation is related to the anisotropies of the stochastic gravitational background. The variation of the correlation with the earth's rotation may be decomposed into harmonics of the earth's period. In section Section V we introduce a set of functions $\gamma_{\ell m}(f)$ which are generalizations of the well-known overlap reduction function $\gamma(f)$ of references [5–8]. These functions characterize the effect of the ℓ, m anisotropy multipole on the m'th harmonic of the detector correlation. The principal result of this paper is to compute these functions for the LIGO pair of detectors. This is done by introducing two special frames of reference, fixed with respect to the earth, in Section VI, and then performing a set of integrations in Section VII. The final integrations are performed and explicit formula for the $\gamma_{\ell m}(f)$ are obtained in Section VIII. This is followed by a short conclusion.

Throughout this paper, c denotes the speed of light and G denotes Newton's gravitational constant.

II. THE STOCHASTIC BACKGROUND

The gravitational wave background may be described in terms of a perturbation to the Minkowski metric of space-time:

$$ds^2 = -c^2 dt^2 + d\vec{x}^2 + h_{ab}(t,\vec{x})dx^a dx^b. \tag{2.1}$$

In transverse traceless gauge, this can be written in the form of a plane wave expansion as

$$h_{ab}(t,\vec{x}) = \sum_A \int_{-\infty}^{\infty} df \int_{S^2} d\hat{\Omega}\, h_A(f,\hat{\Omega}) e^{2\pi i f(t-\hat{\Omega}\cdot\vec{x}/c)} e^A_{ab}(\hat{\Omega}). \tag{2.2}$$

Here $h_A(f,\hat{\Omega})$ is an arbitrary complex function satisfying the relation $h_A(-f,\hat{\Omega}) = h^*_A(f,\hat{\Omega})$. The polarization states are labeled by $A = +, \times$ and $\hat{\Omega}$ is a unit vector on the two-sphere. The wave-vector of the corresponding component of the perturbation is $\vec{k} = 2\pi f\hat{\Omega}/c$. The polarization tensors e^A_{ab} appearing in these relations may be given explicitly. In standard angular coordinates (θ,ϕ) on the two-sphere one may write

$$\hat{\Omega} = \cos\phi\sin\theta\hat{x} + \sin\phi\sin\theta\hat{y} + \cos\theta\hat{z} \tag{2.3}$$

$$\hat{m} = \sin\phi\hat{x} - \cos\phi\hat{y} \tag{2.4}$$

$$\hat{n} = \cos\phi\cos\theta\hat{x} + \sin\phi\cos\theta\hat{y} - \sin\theta\hat{z} \tag{2.5}$$

and then choose

$$e^+_{ab}(\hat{\Omega}) = m_a m_b - n_a n_b \tag{2.6}$$

$$e^\times_{ab}(\hat{\Omega}) = m_a n_b + n_a m_b \tag{2.7}$$

To simplify matters later (but without any loss of generality) we assume that the $\hat{z}$ vector points along the direction of the earth's rotation axis. One can verify by inspection that $\hat{m}$ and $\hat{n}$ are a pair of orthogonal unit-length vectors in the plane perpendicular to $\hat{\Omega}$. It is simple to show that any rotation of the vectors $\hat{m}$ and $\hat{n}$ within the plane that they define simply corresponds to a trivial re-definition of the complex wave amplitudes h_+ and $h_\times$.

To describe a stochastic source, we treat the complex amplitude $h_A(f,\hat{\Omega})$ as a random variable with zero mean value. In this paper, we consider stochastic sources which are *not isotropic*. In principle, such a source has spectral properties which depends upon amplitude and frequency in an arbitrary way. For simplicity, in this paper *we consider only stochastic sources whose directional dependence is frequency-independent*. The dependence of the stochastic background on frequency and direction may be stated in terms of the expectation value of the product of two random variables $h_A(f,\hat{\Omega})$:

$$\langle h^*_A(f,\hat{\Omega}) h_{A'}(f',\hat{\Omega}')\rangle = \delta_{AA'}\delta(f-f')\delta^2(\hat{\Omega},\hat{\Omega}')H(f)P(\hat{\Omega}). \tag{2.8}$$

Here $\delta^2(\hat{\Omega},\hat{\Omega}')$ is a covariant two-dimensional delta-function on the unit two-sphere. For a general stochastic source, the quantity $H(f)P(\hat{\Omega})$ which appears on the right hand side would be an arbitrary function of frequency and direction. However our assumption that the directional dependence is frequency-independent implies that the r.h.s. factors as shown.

III. SPECTRUM OF THE STOCHASTIC BACKGROUND

The function $H(f)$ determines the spectrum of the gravitational radiation. The energy density in gravitational waves is given by

$$\rho_{\rm gw} = \frac{c^2}{32\pi G}\langle \dot h_{ab}\dot h^{ab}\rangle, \tag{3.1}$$

where the overdot denotes a time derivative, and both tensors are evaluated at the same space-time point $(t, \vec{x})$. Substituting the plane wave expansion (2.2) into this formula and using (2.8) yields

$$\langle \dot h_{ab}(t,\vec{x})\dot h^{ab}(t,\vec{x})\rangle = \sum_A \int_{-\infty}^{\infty} df \int_{S^2} d\hat\Omega\, 4\pi^2 f^2 H(f) P(\hat\Omega) e^A_{ab}(\hat\Omega) e^{ab}_A(\hat\Omega). \tag{3.2}$$

Since $\sum_A e^A_{ab} e^{ab}_A = 4$ one has

$$\langle \dot h_{ab}\dot h^{ab}\rangle = 16\pi^2 \int d\hat\Omega P(\hat\Omega) \int_{-\infty}^{\infty} df\, f^2 H(f) = 32\pi^2 \int d\hat\Omega P(\hat\Omega) \int_0^{\infty} df\, f^2 H(f). \tag{3.3}$$

In describing gravitational wave stochastic backgrounds, it is conventional to compare the energy density to the critical energy density $\rho_{\rm critical}$ required (today) to close the universe. This critical energy density is determined by the rate at which the universe is expanding today. Let us denote the Hubble expansion rate today by

$$H_0 = 100\, h_{100}\, \frac{\rm km\, sec^{-1}}{\rm Mpc} = 3.2 \times 10^{-18} h_{100}\, {\rm sec}^{-1} = 1.1 \times 10^{-28} c\, h_{100}\, {\rm cm}^{-1}. \tag{3.4}$$

The value of H_0 is determined by the dimensionless factor of h_{100} which probably lies within the range $1/2 < h_{100} < 1$. The critical energy-density required to just close the universe is

$$\rho_{\rm critical} = \frac{3c^2 H_0^2}{8\pi G} \approx 1.6 \times 10^{-8} h_{100}^2\, {\rm ergs/cm^3}. \tag{3.5}$$

The spectrum of an *isotropic* stochastic gravitational wave background is defined by a dimensionless function of frequency f

$$\Omega_{\rm gw}(f) \equiv \frac{1}{\rho_{\rm critical}} \frac{d\rho_{\rm gw}}{d\ln f}. \tag{3.6}$$

Here $d\rho_{\rm gw}$ is the energy-density in gravitational waves contained within the frequency interval $(f, f+df)$. Using the definition $\Omega_{\rm gw}$ one obtains the relationship between the spectrum $\Omega_{\rm gw}$ and $H(f)$. For $f \ge 0$ one has

$$\Omega_{\rm gw}(f) = \frac{f}{\rho_{\rm critical}} \frac{d\rho_{\rm gw}}{df} = f \frac{8\pi G}{3c^2 H_0^2} \frac{c^2}{32\pi G} 32\pi^2 f^2 H(f) \int d\hat\Omega P(\hat\Omega) = \frac{8\pi^2}{3H_0^2} f^3 H(f) \int d\hat\Omega P(\hat\Omega). \tag{3.7}$$

This formula shows the precise interpretation of $P(\hat\Omega)$. The stochastic background energy density is made of contributions arriving from all directions $\hat\Omega$ on the sky. The actual value of $\Omega_{\rm gw}(f)$ is determined by the average value of $P(\hat\Omega)$; the direction dependence of this function is the same as the direction dependence of the arriving radiation intensity.

For this reason, we *define the multipole moments $p_{\ell m}$* of the stochastic background radiation by the expansion of $P(\hat\Omega)$ in terms of spherical harmonic functions:

$$P(\hat\Omega) \equiv \sum_{\ell m} p_{\ell m} Y_{\ell m}(\hat\Omega) \tag{3.8}$$

where the sum is defined by

$$\sum_{\ell m} \equiv \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} = \sum_{m=-\infty}^{\infty} \sum_{\ell=|m|}^{\infty}. \tag{3.9}$$

In addition, without loss of generality we adopt the convention that the monopole moment is normalized by the condition

$$p_{00} \equiv \sqrt{4\pi} \Rightarrow \int d\hat\Omega P(\hat\Omega) \equiv 4\pi, \tag{3.10}$$

where we assume that the spherical harmonic functions are normalized in the conventional way, so that the integrals of their squares over the unit sphere gives unity. Hence the spectrum of radiation is determined entirely by $H(f)$ since for $f \geq 0$ one has

$$\Omega_{\rm gw}(f) = \frac{32\pi^3}{3H_0^2} f^3 H(f). \tag{3.11}$$

The directionality of the arriving radiation is determined entirely by the function $P(\hat{\Omega})$. Our fundamental assumption here is that the pattern of the intensity of the stochastic background is *fixed in a frame of reference at rest with respect to the cosmological fluid.* In other words, formula (3.8) for $P(\hat{\Omega})$ is expressed in a set of coordinates $x.y, z$ which are fixed with respect to the distant stars. In those coordinates, the multipole moments $p_{\ell m}$ are constants, independent of time. The problem we address in this paper is this: how do we determine, from the data stream of a pair of interferometric detectors which are rotating with the earth, the values of (or bounds on) the multipole moments $p_{\ell m}$?

IV. DETECTION STRATEGY

To determine the multipole moments $p_{\ell m}$ the basic idea is to correlate the outputs of two gravitational wave detectors, and to look for variations of this correlated signal that are harmonics of the earth's rotational frequency. For this purpose, we need to consider the relationship between two different time (or frequency) scales that occur.

The first time scale is that defined by the light travel time ΔT between the two sites. For the remainder of this section, we will assume that the two sites are the Hanford and Livingston LIGO detectors, so that $\Delta T = 10.00$ msec. The second time scale is the period of the earth's rotation about its axis relative to the cosmic frame, $T_e = 8.6 \times 10^4$ sec $= 1$ sidereal day. Because of the enormous disparity between these two time scales, we can define a third time scale, which we will refer to as the averaging time scale, τ. We choose τ in the range

$$\Delta T << \tau << T_e, \tag{4.1}$$

for example $\tau = 30$ sec. It is then possible to examine correlations between the two detectors as a function of time, averaged over periods of length τ. Because τ is much shorter than T_e, the correlation between the two detectors will vary as the earth rotates relative to the fixed cosmological frame, because of the anisotropy in $P(\hat{\Omega})$. On the other hand, because τ is much longer than the light travel time between the two detectors, and because the detectors are sensitive to frequencies $f \approx 1/\Delta T$, there is a significant correlated signal on time scales shorter than τ.

Denote the output of the first detector by

$$s_1(t) = h_1(t) + n_1(t), \tag{4.2}$$

where h_1 is the strain due to the stochastic background and n_1 is the intrinsic noise of the first detector. In similar fashion, the output of the second detector is

$$s_2(t) = h_2(t) + n_2(t). \tag{4.3}$$

Let us use the subscript $i = 1, 2$ to label the detectors, so for example $i = 1$ denotes the Hanford, WA LIGO detector and $i = 2$ denotes the Livingston, LA LIGO detector. The response h_i of detector i to the gravitational radiation is given by

$$h_i(t) = d_i^{ab}(t) h_{ab}(t, \vec{x}_i(t)), \tag{4.4}$$

where the position of detector i's corner station is denoted by $\vec{x}_i(t)$. In this expression, the symmetric traceless tensors $d_i^{ab}(t)$ are given by

$$d_i^{ab}(t) = \frac{1}{2}\left(\hat{X}_i^a(t)\hat{X}_i^b(t) - \hat{Y}_i^a(t)\hat{Y}_i^b(t)\right) \tag{4.5}$$

where the directions of detector i's arms are defined by the unit spatial vectors $\hat{X}_i^a(t)$ and $\hat{Y}_i^a(t)$. Note that *both* $d_i^{ab}(t)$ *and* $\vec{x}_i(t)$ *are functions of time, because the earth rotates with respect to the cosmological rest frame.*

Define quantities which are the Fourier transforms of the signals, evaluated over an interval of one averaging time τ centered at time t:

$$\tilde{s}_i(f, t) = \int_{t-\tau/2}^{t+\tau/2} dt' \, e^{-2\pi i f t'} s_i(t') \text{ for } i = 1, 2. \tag{4.6}$$

These Fourier transforms are easily evaluated. Substituting the plane wave expansion (2.2) into the formula for the strain (4.4) and taking the Fourier transform (4.6) we obtain

$$\tilde{s}_i(f,t) = \sum_A \int d\hat{\Omega} \int_{-\infty}^{\infty} df' e^{2\pi i(f'-f)t} \delta_\tau(f-f') h_A(f',\hat{\Omega}) d_i^{ab}(t) e_{ab}^A(\hat{\Omega}) e^{-2\pi i f'\hat{\Omega}\cdot\vec{x}_i(t)/c} + \text{noise term.} \tag{4.7}$$

In this expression, we have made use of the fact that the averaging time τ is much less than the rotation period of the earth ΔT_e, so that the vectors $X_i^a(t)$, $Y_i^a(t)$, and $\vec{x}_i(t)$ may be treated as constants and taken outside of the time integration in the Fourier transform (4.6). We have also defined the "finite time" approximation to the Dirac delta function

$$\delta_\tau(f) \equiv \int_{-\tau/2}^{\tau/2} dt'\, e^{-2\pi i f t'} = \frac{\sin(\pi f \tau)}{\pi f}, \tag{4.8}$$

which reduces to the Dirac δ-function $\delta(f)$ in the limit $\tau \to \infty$, but has the property that $\delta_\tau(0) = \tau$. The final term on the right hand side of (4.7) is linearly proportional to the noise in detector i.

We now define the "signal"

$$S(t) = \int_{-\infty}^{\infty} df \tilde{s}_1^*(f,t) \tilde{s}_2(f,t) \tilde{Q}(f) \tag{4.9}$$

where $\tilde{Q}(f)$ is an optimal filter function, to be determined. Let us now determine the expectation value of $S(t)$ and show how it incorporates information about the multipole moments of the stochastic background. To find the expected value $S(t)$ we begin by assuming that the noise in each detector has zero mean value, and is uncorrelated with noise and gravitational strain in the other detector. Under these assumptions, we find

$$\begin{aligned}\langle S(t)\rangle = &\int_{-\infty}^{\infty} df \tilde{Q}(f) \sum_A \sum_{A'} \int_{-\infty}^{\infty} df' \int_{-\infty}^{\infty} df'' \int d\hat{\Omega} \int d\hat{\Omega}' e^{-2\pi i(f'-f)t} e^{2\pi i(f''-f)t} \times \\ &\delta_\tau(f-f')\delta_\tau(f-f'') d_1^{ab}(t) d_2^{cd}(t) e_{ab}^A(\hat{\Omega}) e_{cd}^{A'}(\hat{\Omega}') e^{2\pi i(f'\hat{\Omega}\cdot\vec{x}_1(t) - f''\hat{\Omega}'\cdot\vec{x}_2(t))/c} \times \\ &\langle h_A^*(f',\hat{\Omega}) h_{A'}(f'',\hat{\Omega}')\rangle.\end{aligned}$$

We now substitute in the expectation value for the product of the amplitudes (2.8). The integration over f'' is now trivial. In the resulting expression, because $1/\tau$ is much smaller than the "bandwidth" $1/\Delta T$ of the signals, one of the finite-width delta functions δ_τ may be replaced by a Dirac delta function. The integration over f' is then trivial. The other finite-width delta function is then evaluated at zero argument, giving rise to a factor of τ. One thus obtains

$$\langle S(t)\rangle = \tau d_1^{ab}(t) d_2^{cd}(t) \int_{-\infty}^{\infty} df \tilde{Q}(f) H(f) \int d\hat{\Omega} P(\hat{\Omega}) e^{2\pi i f\hat{\Omega}\cdot\Delta\vec{x}(t)/c} \sum_A e_{ab}^A(\hat{\Omega}) e_{cd}^A(\hat{\Omega}) \tag{4.10}$$

where $\Delta\vec{x}(t) = \vec{x}_1(t) - \vec{x}_2(t)$ is the time-dependent separation vector between the two interferometer sites.

Not surprisingly, this previous expression can be easily simplified for the isotropic case $P(\hat{\Omega}) = 1$. In this instance, the sum over polarizations and integral over directions can be performed explicitly, yielding ($8\pi/5$ times) a time-independent function of frequency known as the overlap reduction function $\gamma(f)$. This overlap reduction function is given by

$$\gamma(f) \equiv \frac{5}{8\pi} d_1^{ab} d_2^{cd} \int_{S^2} d\hat{\Omega}\, e^{2\pi i f\hat{\Omega}\cdot\Delta\vec{x}/c} \left(e_{ab}^+(\hat{\Omega}) e_{cd}^+(\hat{\Omega}) + e_{ab}^\times(\hat{\Omega}) e_{cd}^\times(\hat{\Omega}) \right). \tag{4.11}$$

Notice that in (4.11) the dependence of the positions and orientations of the detectors upon time t is *not* shown; this is because $\gamma(f)$ depends only upon the *relative* positions and orientations, which is time (or earth-position) independent. Thus, in the case of an isotropic stochastic background, one finds

$$P(\hat{\Omega}) = 1 \Rightarrow \langle S(t)\rangle = \frac{8\pi}{5} \tau \int_{-\infty}^{\infty} df \tilde{Q}(f) H(f) \gamma(f). \tag{4.12}$$

This is equation (30) of reference [8]. In the present paper, we are most interested in the anisotropic case where $P(\hat{\Omega})$ varies with direction. In this case, the time variation of the tensors $d_i^{ab}(t)$ and $\Delta\vec{x}(t)$ will provide a time-dependent variation of the signal $S(t)$.

V. ROTATION HARMONICS

Because the rotation of the earth is periodic with period T_e and angular frequency $\omega_e = 2\pi/T_e$ the expected signal (4.10) varies with the same period. It can therefore be represented by the Fourier series

$$\langle S(t)\rangle = \sum_{m=-\infty}^{\infty} S_m e^{im\omega_e t}. \tag{5.1}$$

Because the signal is real, the amplitudes of the different harmonics satisfy $S_m = S^*_{-m}$. The amplitudes are obtained by inverting the Fourier series:

$$S_m = \frac{1}{T_e}\int_0^{T_e} dt\, e^{-im\omega_e t}\langle S(t)\rangle. \tag{5.2}$$

The harmonic amplitudes S_m are the (at least in principle) observable quantities on which any data analysis must be based.

Because we have assumed that the z-axis of our (cosmic) coordinate system points along the direction of the earth's axis, the m'th rotation harmonic can only result from anisotropies whose phase varies with angle ϕ as $\exp(im\phi)$. These are the anisotropies associated with the $Y_{\ell m}$. Hence

$$S_m = \frac{8\pi}{5}\tau \int_{-\infty}^{\infty} df \tilde{Q}(f) H(f) \sum_{\ell=|m|}^{\infty} P_{\ell m}\gamma_{\ell m}(f) \tag{5.3}$$

The functions $\gamma_{\ell m}(f)$ are generalizations of the overlap reduction function $\gamma(f)$, which express the (frequency-dependent) contribution of the ℓ'th multipole moment to the m'th harmonic of the signal, with respect to the Earth's rotation. These are given by

$$\gamma_{\ell m}(f) = \frac{5}{8\pi}\frac{1}{2\pi}\int_0^{2\pi} d\alpha\, e^{-im\alpha} d_1^{ab}(\alpha) d_2^{cd}(\alpha) \int d\hat\Omega\, Y_{\ell m}(\hat\Omega) e^{2\pi i f \hat\Omega\cdot\Delta\vec{x}(\alpha)/c} \sum_A e^A_{ab}(\hat\Omega) e^A_{cd}(\hat\Omega). \tag{5.4}$$

In this expression, the angle of rotation of the earth about its axis (measured from some arbitrary fiducial point) is denoted by $\alpha \in [0, 2\pi)$ so $\alpha = \omega_e t + \text{constant} \pmod{2\pi}$. The "time-dependent" quantities d_i^{ab} and $\Delta\vec{x}$ may equivalently be expressed as functions of α.

The problem at hand is now a mathematical one – to calculate the functions $\gamma_{\ell m}(f)$ which are generalizations of *the overlap reduction function* $\gamma(f)$. For the monopole moment ($\ell = m = 0$) it is easy to see that the integrand above is independent of earth-position α because the overlap reduction function (4.11) only depends upon the relative orientations of the detectors, which is α-independent, giving

$$\gamma_{00}(f) = (4\pi)^{-1/2}\gamma(f). \tag{5.5}$$

In the next parts of this paper, we will show how to evaluate the other $\gamma_{\ell m}$.

Our first task is to evaluate the integrals that appear in (5.4). The product $d_1^{ab}(\alpha)d_2^{cd}(\alpha)$ is a quartic polynomial in $\sin\alpha$ and $\cos\alpha$. One approach would be to attempt to perform the integral over $\hat\Omega$, to obtain the resulting function of α, and then to evaluate the integral over α. However this approach is rather cumbersome.

A more promising method is to consider the projector onto the plane perpendicular to $\hat\Omega$, which may be calculated in terms of the vectors defined by (2.4) and (2.5):

$$Q_{ab} = \delta_{ab} - \hat\Omega_a\hat\Omega_b = \hat m_a \hat m_b + \hat n_a \hat n_b. \tag{5.6}$$

A couple minutes of algebra starting with (2.6) and (2.7) quickly establishes the identity

$$\sum_A e^A_{ab}(\hat\Omega) e^A_{cd}(\hat\Omega) = Q_{ac}Q_{bd} + Q_{ad}Q_{bc} - Q_{ab}Q_{cd}. \tag{5.7}$$

We then define the set of integrals

$$C_{abcd}(\alpha) = \int d\hat\Omega\, Y_{\ell m}(\hat\Omega) e^{2\pi i f\hat\Omega\cdot\Delta\vec{x}(\alpha)/c}\hat\Omega_a\hat\Omega_b\hat\Omega_c\hat\Omega_d. \tag{5.8}$$

The desired integrals can then be expressed in terms of this quantity. For convenience, we introduce a symbol to handle the contractions that occur. This is a constant tensor defined by

$$\Theta^{pqrs}_{abcd} = 2\delta_{ac}\delta_{bd}\delta^{pq}\delta^{rs} - 4\delta_{ac}\delta^{pq}\delta_b{}^r\delta_d{}^s + \delta_a{}^p\delta_b{}^q\delta_c{}^r\delta_d{}^s. \tag{5.9}$$

Making use of the fact that each of the d_i^{ab} is symmetric in its tensor indices, and traceless, we may then write

$$\gamma_{\ell m}(f) = \frac{5}{8\pi}\frac{1}{2\pi}\int_0^{2\pi} d\alpha\, \mathrm{e}^{-im\alpha} d_1^{ab}(\alpha) d_2^{cd}(\alpha)\Theta^{pqrs}_{abcd} C_{pqrs}(\alpha). \tag{5.10}$$

In order to evaluate C_{abcd} it is convenient to introduce some additional coordinate systems.

VI. COORDINATE FRAMES

The vectors being used in this calculation are three-dimensional spatial vectors in flat Cartesian R^3. Up to this point, we have been using a coordinate system which is fixed with respect to the cosmological fluid, and in which the spatial pattern of the perturbations of the stochastic background is assumed to be time-independent. This frame of reference is the "unprimed" frame; vectors expressed with respect to these cosmic coordinates have unprimed indices. We have also assumed (without any loss of generality) that the z-axis of this cosmic frame points along the direction of the earth's rotation axis.

At this point, for calculational purposes, it is convenient to consider two additional coordinate systems. Thus, a given vector V may be expressed in terms of its components in three different frames:

$$\begin{aligned} \text{Cosmic Frame}: &\quad V^a \\ \text{Earth Frame}: &\quad V^{\bar a} \\ \text{Computational Frame}: &\quad V^{a'} \end{aligned}$$

The "earth frame" is a coordinate system fixed to the earth, in which the third (z-coordinate) points along the axis of the earth's rotation, in the direction of the North pole. Components of vectors in this frame are denoted with "barred" indices. The second of these new coordinate systems will be referred to as the "calculational" coordinate system. In this frame, the components of vectors are "primed". This frame is fixed with respect to the earth, and has its third (z-coordinate) pointing along the line between the two gravitational-wave detectors.

The relationship between components of vectors in these three coordinate frames may be written as matrix equations. Each of the matrices which appears is a special case of a rotation matrix which may be parametrized by Euler angles. Throughout this paper, we use the Euler angle conventions given by equations (4.83-6) of Afkin [9] which are also the conventions used in equations (4.5) and (4.43) of Rose [10]. It is convenient to define a pair of rotations about the z and y axes respectively, by

$$\mathbf{R}_z(\alpha) \equiv \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{R}_y(\beta) \equiv \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix} \tag{6.1}$$

The most general possible rotation may be parametrized by Euler angles and is defined by the matrix $\mathbf{R}(\alpha,\beta,\gamma) = \mathbf{R}_z(\gamma)\mathbf{R}_y(\beta)\mathbf{R}_z(\alpha)$. Note that the boldface symbols here denote 3×3 square matrices.

The matrix which relates components of vectors in the cosmic and earth frames is simply rotation through angle α about the z-axis:

$$X^{\bar a} = R^{\bar a}{}_a X^a \qquad R^{\bar a}{}_a = \mathbf{R}(\alpha,0,0) = \mathbf{R}_z(\alpha) \tag{6.2}$$

where the first index on R labels rows and the second index labels columns, so that the operation appearing in the previous equation is ordinary multiplication of a column vector on the right by a square matrix on the left. Note that the angle $\alpha = \omega_e t$ varies with time.

Without loss of generality, assume that the freedom to choose the $\bar x$- and $\bar y$-axis in the Earth frame has been used to ensure that in this frame the separation vector $\Delta x^{\bar a}$ between the two detector sites has no $\bar y$-component. Using the two LIGO sites as an example, the Earth-frame $\bar x$-axis would point out from the center of the earth at an angle 38.6881° East of the 0° line of longitude (Greenwich, England). In this frame, the coordinates of the two detector sites and the detector arms directions are