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Analysis of the Pendular and Pitch Motions of a Driven Three-Dimensional Pendulum

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Abstract

A three-dimensional pendulum, modeled after the Laser Interferometer Gravitational-wave Observatory's suspended optics, was constructed to investigate the pendulum's dynamics due to suspension point motion. In particular, we were interested in studying the pendular-pitch energy coupling. Determination of the pendular's Q value was made from frequency domain analysis of the experimental data. However, the pitch's Q value had to be calculated using a numerical simulation.

1 Introduction

The Laser Interferometer Gravitational-wave Observatory (LIGO) detector [1] is a Michelson interferometer designed to detect strains of space-time on the order of 10^{-21} or less. The lengths of the two interferometric arms are configured so that the light beams returning from the end mirrors interfere destructively at the beam splitter when a gravitational wave is not present. Thus, when a gravitational wave coming to the earth stretches space along one arm and compresses space along the other arm, the condition of the destructive interference breaks and the gravitational wave signal can be detected as the temporal characteristics of the light leaking through the beam splitter. To increase the sensitivity, the arms are configured as optical cavities so that the distance that light travels in the respective arms is effectively increased. To detect a gravitational wave signal in this fashion, it is essential that the mirrors in the interferometer act as free masses in the frequency range in which gravitational wave signal is to be detected. To this end, the mirrors forming the arm cavities and the beam splitter are suspended and their positions are actively controlled. The optics are placed on vibration-isolation mechanisms to isolate them from seismic disturbance, and the entire interferometer is placed in a vacuum chamber to minimize wavefront distortion of the light due to scattering by molecules.

The dynamics of the suspended mirror is an interesting subject for undergraduates. In our previous paper [2], we modeled LIGO's suspended optics as a one dimensional damped driven oscillator, and studied how the motion of the suspension point was transferred to the suspended mass. As a continuation of this work, we wanted to improve our understanding of the optics' dynamics by modeling an optic as a three-dimensional disk. The pendular and yaw motions are directly excited by the suspension point motion, so their analyses are relatively easy. However, the pitch motion cannot be directly excited from the suspension point, and therefore, excitation of the pitch motion by seismic disturbance (which moves the suspension point) is due to a coupling of the pendular and pitch motions. Suppression of pitch motion is important in the LIGO detector to reduce exciting higher order modes in the arm cavities. In this study, we set up a free hanging model optic and recorded the pendular and pitch motions due to an induced suspension point motion. The pendular motion was compared to analytical results. However, it is difficult to describe the pitch motion analytically due to the pendular-pitch energy transfer. Therefore, we used a numerical simulation to calculate the pitch motion, which we then compared to our experimental data.

2 LIGO suspended optic

Fig. 1 illustrates the LIGO suspended Small Optic [3] schematically. A disk optic of 7.5 cm in diameter and 2.5 cm in thickness is suspended by a single wire whose ends are attached to a suspension tower with a separation of 3.5 cm. The wire is looped around the disk via two wire-standoff attached to both sides as shown. To make the suspended optic stable, the wire-standoffs are attached slightly above the center of mass, which is at the geometric center of the disc. In Fig. 1, R_1 is the distance from the vertical line going through the geometric center of the optic to the point where the wire is attached to the tower (the suspension point), R_2 is the distance from the same vertical line to the point where the wire is attached to the optic, l is the vertical distance from the suspension point to the optic's center of mass, b is the vertical distance between the point where the wire is attached to the optic and the horizontal line going through the optic's center of mass, and R is the radius of the optic.

Pendular, pitch and yaw degrees of freedom (DOF) of the suspended optic are important from the viewpoint of forming a cavity. Here the pendular DOF represents translational motion of the optic's center of mass normal to its reflective surface; the pitch DOF represents rotational motion around the horizontal axis going through the geometric center of the disk and parallel to the reflective surface; and the yaw DOF represents rotational motion around the vertical axis going through the disk's geometric center. The tower is a rigid structure and, therefore, directly channels the table motion to the suspension point. The planar motion of the suspension point causes the pendular motion of the optic. The yaw motion is due to the separation of the optic's wires at the suspension point by a distance of $2R_1$. Energy is transferred from the pendular motion to

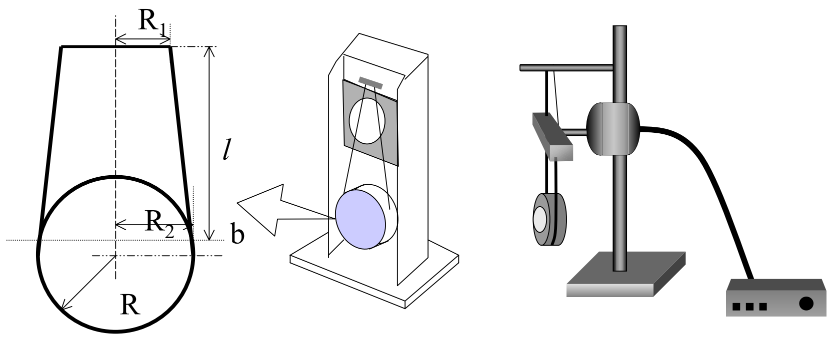

Figure 1: LIGO small optic (left), LIGO small optic with suspension (center), model pendulum (right)

pitch motion because the center of mass of the disk is below the axis around which the disk can rotate vertically (in other words, there is a finite arm length between the point where the wire exerts a force on the disk and the gravity exerts the gravitational force); however, pitch cannot be directly excited from the suspension point.

The equations of motion governing these three motions are written in the following form.

$$\ddot{x} + \gamma_x \dot{x} + \omega_x^2 x = \omega_x^2 (x_{sp} + b\theta) \tag{1}$$

$$\ddot{\theta} + \gamma_\theta \dot{\theta} + \omega_\theta^2 \theta = \frac{\omega_\theta^2}{l+b}(x - x_{sp}) \tag{2}$$

$$\ddot{\phi} + \gamma_\phi \dot{\phi} + \omega_\phi^2 \phi = \omega_\phi^2 \phi_{sp} \tag{3}$$

where x, θ, and ϕ are translational and rotational displacement representing the pendular, pitch and yaw DOF, respectively; x_{sp} and ϕ_{sp} are the translational and rotational displacement of the suspension point in the pendular and yaw DOF; γ_x, γ_θ, γ_ϕ are the damping coefficients. ω_x, ω_θ, ω_ϕ are the natural (uncoupled) frequencies for pendular, pitch and yaw motion, respectively, given as

$$\omega_x^2 = g/l \tag{4}$$

$$\omega_\theta^2 = \frac{mg}{I_\theta l} b(l+b) \tag{5}$$

$$\omega_\phi^2 = \frac{mg}{I_\phi l} R_1 R_2 \tag{6}$$

where I_θ and I_ϕ are the moments of inertia around the horizontal and vertical lines going through the geometric center of the disk. Note that ω_ϕ depends on b via R_2 ($R_2 = \sqrt{R^2 - b^2}$). When dissipation is low, the eigen frequencies Ω for the pendular, pitch and yaw DOF can be written in the following form.

$$\Omega_{x,\theta}^2 \approx \frac{g}{2lI_\theta}[I_\theta + mb(l+b) \mp \sqrt{[I_\theta - mb(l+b)]^2 + 4mb^2 I_\theta}] \tag{7}$$

$$\Omega_\phi^2 \approx \frac{mg}{I_\phi l} R_1 R_2 \tag{8}$$

3 Experiment

3.1 The Model Optic

We wanted our model to represent the LIGO Small Optic. A wooden disk was cut to approximately the dimensions of the LIGO Small Optic [3]. We mimicked the LIGO's suspension mechanism by wrapping a wire under the disk and attaching it to the sides of the disk; however, we did not use a standoff, so the exact value for b was unknown. The wire was suspended from a wooden suspension block with predrilled holes for the wire approximately the same spacing as for the LIGO suspension point. The suspension block had a hole drilled in the side to fit a mechanical oscillator pin that would be used to drive the suspension point. Since the LIGO tower is a rigid structure, its transfer function can be approximated as 1; therefore, we directly excited the suspension point instead of the table.

This hole was placed off-center so that we could induce a yaw motion in addition to the pendular motion. To mimic the suspension point motion, we used a mechanical oscillator controlled by a function generator. The mechanical oscillator provided a sinusoidal disturbance to the suspension block. Since the suspension block and disk were too heavy for the mechanical oscillator pin to hold, we tied the suspension block to a horizontal rod attached to the stand above the block. The disks horizontal and vertical axes of rotation were off the center of mass. Consequently, our disk experienced both pendular-pitch and pendular-yaw coupling whereas the LIGO optic is designed to only have pendular-pitch coupling.

3.2 Data Acquisition

We measured the disk's motions simultaneously for all three degrees of freedom. To determine the pitch and yaw motion, a laser was pointed at a mirror attached to the front surface of the disk and the light was reflected onto a measurement plane. To measure the pendular motion, reflective material was placed on a side of the disk at the center and a light was used to illuminate the reflective material. The light from the reflective material hit a mirror placed to the side of the disk and was reflected onto the measurement plane. Since we were only concerned with the planar motion of the center of mass, measuring the motion from the side of the disk was adequate. The reflected light from the mirror on the disk surface indicating pitch and yaw motion and the light from the reflecting material on the side of the disk were captured by the same CCD camera so that we could record all the three DOF simultaneously (Fig. 2) We used a LabView program to take the pictures at 300 ms intervals, which was limited by the frame rate of the CCD camera. A VisionBuilder program was written to extract the relevant motion of the disk and put the results in Excel spreadsheet format. From Excel, we were able to calculate the pendular, pitch and yaw motions of the disk in both the time and frequency domains.

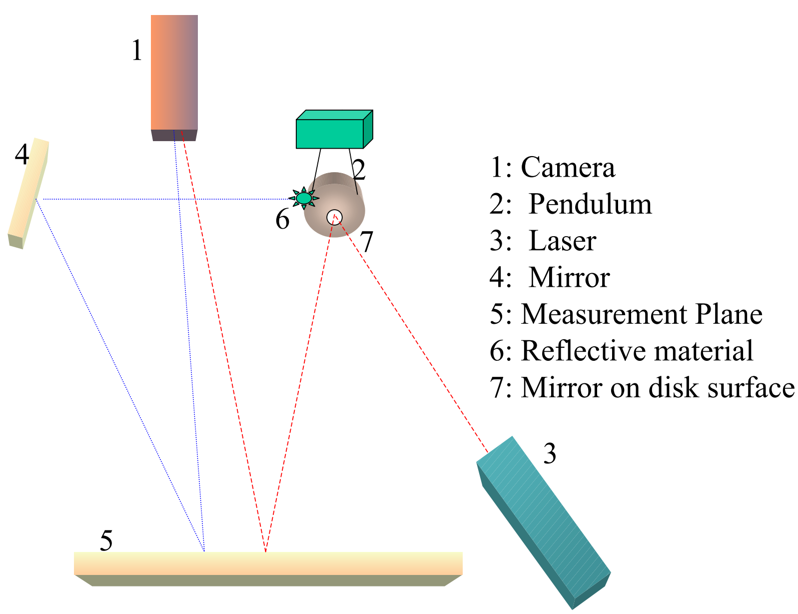

Figure 2: Experimental setup

4 Results and discussion

4.1 Experimental procedures

We carried out two sets of experiments. In the first set, we drove the suspension point to excite the model pendulum's pendular, pitch and yaw degrees of freedom and then removed the driving force after the pendulum was sufficiently excited (called the ringdown experiment). To determine the eigenfrequencies of pendular, pitch and yaw DOF, we analyzed the results in the frequency domain. In the second sets of experiment, we drove the suspension point at various frequencies and measured the response of the pendulum.

Fig. 3 shows the power spectra in the three DOF obtained in the ringdown experiment. To excite each DOF selectively, we drove the suspension point at a frequency close to the eigenfrequency of each DOF. Thus we drove the suspension point at 1 Hz to excite the pendular and yaw DOF, and at 3.5 Hz to excite the pitch DOF. Note that the eigenfrequency of the pendular DOF is close to the actual LIGO's small optics because the wire length is similar, but those of the pitch and yaw DOF are substantially different from the LIGO's case because the wire holds the disk in a different way. From the observed peaks, the eigenfrequencies of pendular, pitch and yaw DOF can be determined as 1.05 Hz, 3.20 Hz, and 1.34 Hz, respectively. Here the the eigenfrequency of the pitch DOF was higher than the Nyquist frequency resulting from the above-mentioned data acquisition interval of 300 ms. Thus, the pitch peak observed in Fig. 3 represents an alias. Following the theory of digital sampling [5] we determined the pitch eigenfrequency as [the sampling frequency of 3.33 Hz] - [the alias peak frequency of 0.13 Hz]=3.20 Hz. A brief description of aliasing is given in the appendix. We directly measured this pitch eigenfrequency by an independent experiment in which we counted the number of pitch oscillations over a fixed duration.

By fitting a theoretical transfer function to the measured frequency response, the pendular Q can be determined as 28 (Fig. 5). The pitch spectrum is good enough to read the peak frequency but not to determine the Q. Because the pitch peak in the frequency domain was small compared to the noise, we could not use the same method to determine the pitch Q value. So, we decided to use the pitch Q as a fitting parameter in a numerical simulation.

4.2 Numerical simulation using End-to-end

To determine the pitch Q value numerically, we used a simulation package called the end-to-end model or e2e, developed by LIGO Laboratory at the California Institute of Technology. E2e is a visual programming language that simulates the LIGO detector in the time domain. Programs are built using lower level modules, called boxes, which simulate a particular aspect of the detector; the lowest level box is referred to as a primitive. The susp3Dmass primitive simulates the motion of a three-dimensional pendulum and can be driven at the suspension point. The primitive allows changes in the physical dimensions of

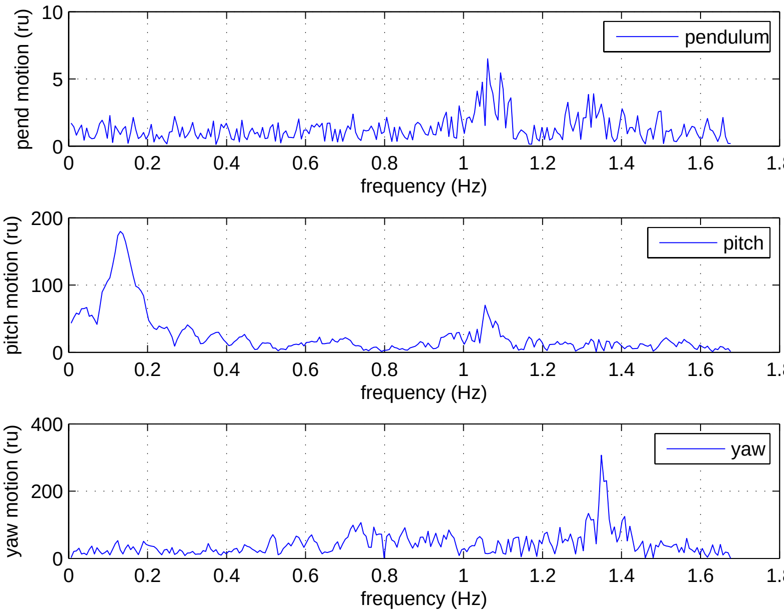

Figure 3: Ringdown power spectra

the pendulum; i.e. mass, length, b, and Q values.

To compute the motion, it is necessary to know b. Since we did not use wire standoffs, the value of b is unclear from the visual examination of the model optic setup. Using the pitch and yaw resonant frequencies, which are both dependent upon b and were measured in the ringdown experiment, we were able to determine b. Using eqs. (5) and (6), we calculated the pitch and yaw frequency as functions of b. The results are shown in Fig. 4, which indicate that when b=1.4 cm, the calculated resonant frequencies of pitch and yaw are both closest to the experimentally determined values. We verified this value by directly measuring the location of the center of mass of the disk in the following manner. We placed the disk on a small cylindrical object so that the disk balanced parallel to the table across the pitch axis. We noted the distance between the axis denoted by the location of the cylindrical object and the point where the string was attached to the disk. This method of measuring b also resulted in b=1.4 cm.

4.3 Determination of the Pitch Q value

We put the dimensions of our disk and the values for b and pendular Q determined as described above into the e2e simulation, leaving the pitch Q as a free

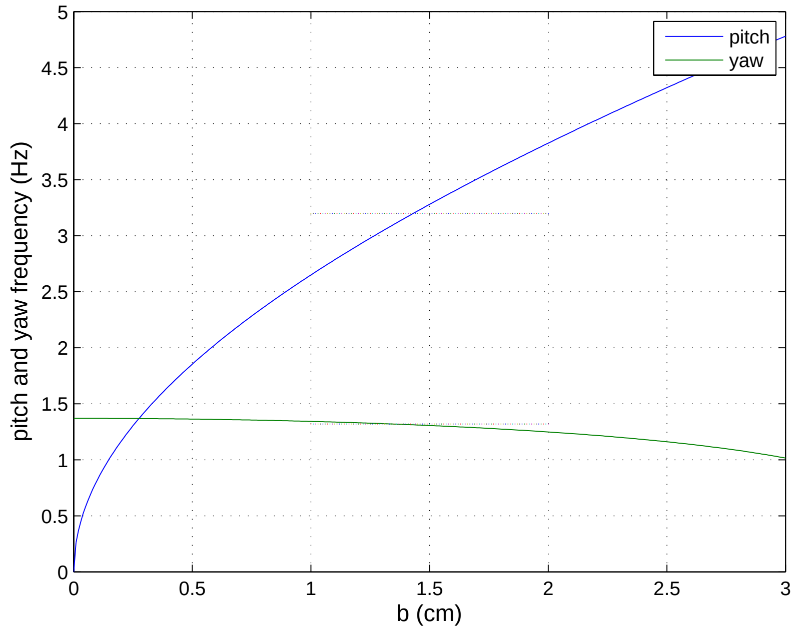

Figure 4: Determination of b

parameter. The Q values of the pendular and pitch degrees of freedom affect the pitch motion in the following way. Suspension point motion due to seismic disturbance transmitted through the vibration-isolation mechanism excites the suspended optic's pendular DOF. Because the axis of vertical rotation does not go through the center of mass (it goes through the two points where the wire exerts forces on the left and right sides of the disk), pendular motion exerts torque around the axis via the gravitational force acting on the center of mass. This excites the disk's motion in the pitch DOF. Because the resonant frequency of the pitch DOF (3.20 Hz) is higher than that of the pendular DOF (1.05 Hz), this excitation of pitch takes place on the blue-side tail of the pendular spectrum. The amplitude of the pendular DOF is determined by the width of the pendular spectrum or the pendular Q, and the susceptibility of the pitch DOF to the pendular excitation depends on the pitch Q value. Thus the spectrum of the disk's pendular motion has two peaks: one at the pendular resonance and the other at the pitch resonance, and the relative height of the two peaks agree with the experiment only when the Q values of the two DOF are the same as experiment.

We ran the program with swept-sine input to the suspension point. We graphed the resulting data in comparison to the experimental data for different values of the pitch Q; consequently, a Q value of 40 for pitch matched the results from our experimental data (Fig. 5).

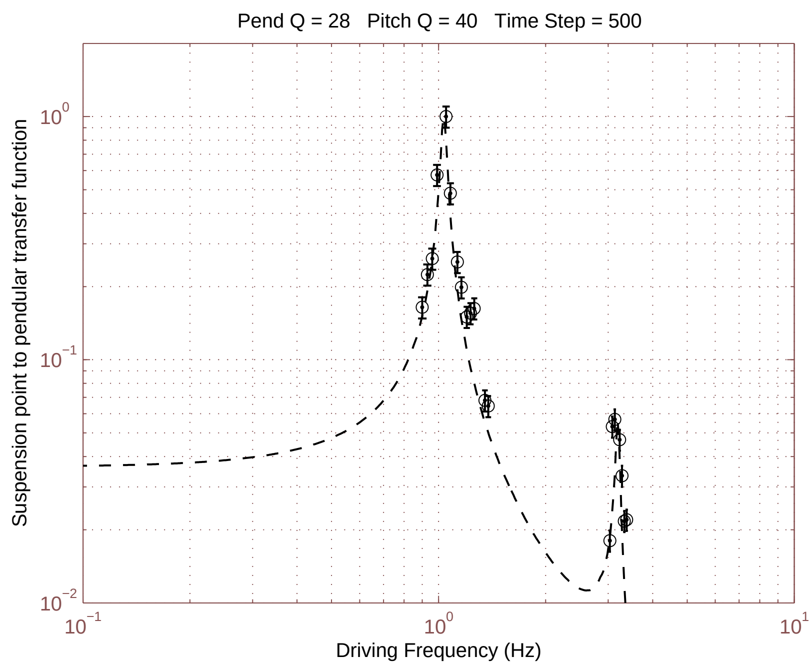

Figure 5: Comparison of simulation and measurement

5 Conclusion

We set up a model LIGO small optic and simulated table motion by driving the model at its suspension point. We investigated the pendular, pitch and yaw motions and determined the resonant frequencies (1.05 Hz, 3.20 Hz, and 1.34 Hz respectively). Analysis of the pendular and yaw motions followed directly from the respective equations of motion; however, due to the pendular-pitch coupling, the pitch analysis was done using LIGO's simulation package, e2e.

The dimensions of the model and experimentally determined pendular Q value of 28 were given to the simulation to determine the pitch Q. A pitch Q value of 40 agreed with the experimental spectrum. The combined effect of the low Q values and nearby resonant frequencies explains the pendular-pitch coupling, which is the only mechanism for exciting the pitch motion in a free hanging pendulum.

6 Acknowledgment

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7 Appendix

We briefly discuss the effect of digital sampling when the sampling frequency is lower than the signal frequency. We focus on the present situation where the sampled signal decays exponentially and the total sampling time is much greater than the decay time. More detailed description can be found in ref. [6].

We assume the original analog signal is of the form:

$$f(t) = \begin{cases} Acos(2\pi f_0 t)e^{-t/\tau} & t > 0 \\ 0 & t < 0 \end{cases}$$

where A is the amplitude, f_0 is the frequency, and τ is the time constant of the original signal.

With the use of the so-called 'comb' function and 'rectangle' function, the sampling function $S(t)$ can be written in the following form:

$$S(t) = \frac{1}{t_s} comb(\frac{t}{t_s}) rect(\frac{t - Nt_s/2}{Nt_s}) \tag{9}$$

where t_s is the sampling interval and Nt_s is the total sampling time resulting from N samples, and the comb and rectangle functions are given as,

$$comb(\frac{t}{t_s}) = \sum_{n=-\infty}^{\infty} \delta(t - nt_s) \tag{10}$$

$$rect(t) = \begin{cases} 0, & |t| > 1/2 \\ 0.5, & t = 1/2 \\ 1, & |t| < 1/2 \end{cases}$$

With these expressions, the Fourier transform of the sampled data F_{SAMP}, which is the product of the signal and the sampling function, can be written in the following form:

$$F_{SAMP}(f) = \mathbf{F}(A\cos(2\pi f_0 t) * \mathbf{F}(e^{-t/\tau}) * \mathbf{F}(\frac{1}{t_s} comb(\frac{t}{t_s})) * \mathbf{F}(rect(\frac{t - Nt_s/2}{Nt_s})) \tag{11}$$

Here "*" denotes the convolution operation, and the boldface F denotes the Fourier transform operations. Of the four terms on the right-hand side of this expression, the exponential decay and the rectangle function contribute to broadening the spectrum, and the cosine term and the comb function contribute to aliasing the peak frequency. Let us call the product of the two terms that contribute to broadening F_B and the product of the other two terms F_{PEAK}.

When the total sampling time is much greater than the decay time;i.e., $Nt_s >> \tau$ the magnitude and phase of F_B take the following forms:

$$|F_B| \cong \frac{1}{\sqrt{[1+(2\pi f\tau)^2]}} \tag{12}$$

$$tan\Phi \cong -2\pi f\tau \tag{13}$$

From this equation, we see that the width of the transformed signal is of order $1/\tau$.

By performing the convolution, F_{PEAK} can be put in the form of

$$F_{PEAK}(f) = A[\sum_{n=-\infty}^{\infty}[\delta(f - f_0 - \frac{n}{t_s}) + \delta(f + f_0 - \frac{n}{t_s})]] \tag{14}$$

Using this, the transform of the sampled data is given by

$$F_{SAMP}(f) = F_B(f) * F_{PEAK}(f) = A[\sum_{n=-\infty}^{\infty}[F_B(f - f_0 - \frac{n}{t_s}) + F_B(f + f_0 - \frac{n}{t_s})]] \tag{15}$$

This equation indicates that multiple peaks are associated with the signal, allowing us to draw several conclusions: (a) When the frequency f_0 satisfies $0 < f_0 < 1/(2t_s)$ (the Nyqist condition), f_0 clearly satisfies $(2n)/(2t_s) < f_0 + n/t_s < (2n+1)/(2t_s)$;, i.e., the first term in eq. (15) ensures that every other interval ($-1/t_s$ to $-1/2t_s$, 0 to $1/2t_s$, $1/t_s$ to $3/2t_s$, etc.) includes a copy of the peak corresponding to f_0. Further, since $(2n-1)(2t_s) < n/t_s - f_0 < n/t_s$, the second term in eq. (15) ensures that there is a copy in the remaining intervals ($-3/2t_s$ to $-1/t_s$, $-1/2t_s$ to 0, $1/2t_s$ to $1/t_s$, etc.) Therefore, only the interval $f = 0$ to $1/2t_s$ need to be considered and there will be precisely one copy of the function $F_B(f)$ in that interval (from the first term in eq. (15) with $n = 0$). (b) When f_0 satisfies $1/2t_s < f_0 < 1/t_S$ (the condition of undersampling), the sampling rate is too short to faithfully reproduce the sampled signal. It is again the case that every interval n/t_s to $(n + 1/2)t_s$ contains the same information. But in this case, the peak that appears in the interval 0 to $1/2t_s$ is from the second term in eq. (15) and occurs at the frequency $f = 1/t_s - f_0$. Referring to the middle plot of Fig. 3, we see a peak at 0.13 Hz, which is an alias of the pitch resonance. Thus the pitch resonance can be computed as 3.33 Hz (the sampling frequency) - 0.13 Hz (alias) = 3.2 Hz.

Thus when the sampling frequency is lower than the signal frequency, the location of resonance peak change in accordance with eqns. (12) and (15).

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