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# Doppler-Induced Dynamics of Fields in Fabry-Perot Cavities with Suspended Mirrors [1]

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## Abstract

The Doppler effect in Fabry-Perot cavities with suspended mirrors is analyzed. Intrinsically small, the Doppler shift accumulates in the cavity and becomes comparable to or greater than the line-width of the cavity if its finesse is high or its length is large. As a result, damped oscillations of the cavity field occur when one of the mirrors passes a resonance position. A formula for this transient is derived. It is shown that the frequency of the oscillations is equal to the accumulated Doppler shift and the relaxation time of the oscillations is equal to the storage time of the cavity. Comparison of the theory with the experiment is discussed and applications of the formula for measurements of the mirror velocity is described.

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Fabry-Perot cavities with length of several kilometers are utilized in laser gravitational wave detectors such as LIGO [1]. The mirrors in these Fabry-Perot cavities are suspended from wires and therefore are free to move along the direction of the beam propagation. Ambient seismic motion excites the mirrors, causing them to swing like pendula with frequencies of about one hertz and amplitudes of several microns. To maintain the cavity on resonance the Pound-Drever locking technique is used [2]. During lock acquisition the mirrors frequently pass through resonances of the cavity. As one of the mirrors approaches a resonant position the light in the cavity builds up. Immediately after the mirror passes a resonance position, a field transient in the form of damped oscillations occurs. The structure of the transient is universal - it does not depend on the initial conditions for the field in the cavity or the trajectories of the mirrors. It only depends on the cavity length, its finesse and the relative velocity of the mirrors. Thus, careful examination of the transient reveals useful information about the cavity properties and the mirror motion.

The oscillatory transients have been observed in the intensity of the transmitted light by An et al [3]. They showed that the period of oscillations is related to the mirror velocity. The transients also have been observed in the Pound-Drever locking signal by Camp et al [4] in connection with the cavity lock acquisition. However, the physics of the oscillations, which appear in the transient, so far has not been fully understood. In this paper a brief theory of the transient is presented (a more detailed account is given in [5]). It is shown that the oscillations are due to the Doppler shift of light circulating in the cavity and a simple formula for the transient is derived. The predictions based on the formula are compared with the data from the 40m Fabry-Perot cavity of the Caltech prototype interferometer.

The Doppler shift in Fabry-Perot cavities can be briefly described as follows. Consider a Fabry-Perot cavity with the length $L$ and the propagation time $T$ ($T = L/c$) as shown in Figure 1. Assume, for simplicity, that one of the mirrors (input mirror) is at rest and the other (end mirror) is freely swinging. Let the trajectory of this mirror be $x(t)$. As the light reflects off of the moving mirror its frequency, $\omega$, is Doppler-shifted, and the shift in one reflection is

$$\delta\omega = -\frac{2v}{c}\omega, \tag{1}$$

where $v = dx/dt$ is the instantaneous velocity of the mirror. Subsequent reflections make the Doppler shift add up, causing it to accumulate with time.

A suspended mirror in gravitational wave detectors moves very little. Its

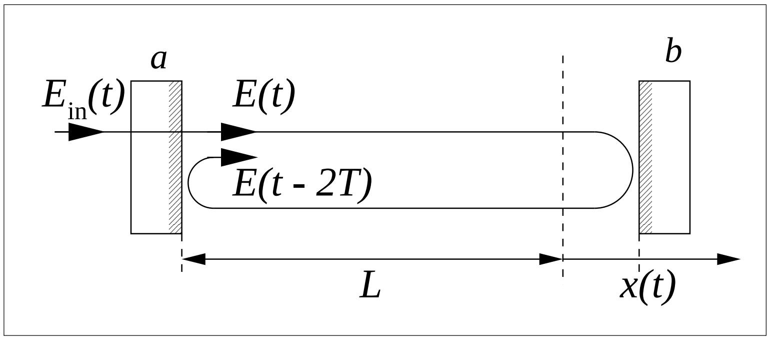

Figure 1: A schematic picture of the field circulating in a Fabry-Perot cavity

r.m.s. velocity is typically of the order of a few microns per second. The corresponding Doppler shift is of the order of a few hertz, which is very small compared to the laser frequency $2.82 \times 10^{14}$ Hz for an infra-red laser with wavelength $\lambda = 1.06\mu$m. However, the line-width of the long Fabry-Perot cavities of gravitational wave detectors is also very small, typically of the order of 100 Hz. Therefore, the small Doppler shift as it accumulates with time can easily exceed the line-width.

The characteristic time scale for light to remain in the cavity is the storage time of the cavity, which we define here as $1/e$ amplitude folding time:

$$\tau = \frac{2T}{|\ln(r_a r_b)|}, \tag{2}$$

where $r_a$ and $r_b$ are the reflectivities of the input and the end mirror respectively. The Doppler shift accumulated within the storage time is

$$\delta\omega \frac{\tau}{2T} = -\omega \frac{v\tau}{cT}. \tag{3}$$

It is comparable to the line-width of the cavity if the relative velocity of the mirrors is comparable to the critical velocity defined as

$$v_{\rm cr} = \frac{\lambda}{2\tau\mathcal{F}} \approx \frac{\pi c\lambda}{4\mathcal{F}^2 L}, \tag{4}$$

where $\mathcal{F}$ is the finesse of the cavity. Thus the Doppler effect becomes significant if the time it takes for a mirror to move across a width of a resonance is comparable to the cavity storage time. The critical velocity, $v_{\rm cr}$, makes it possible to distinguish between fast ($v > v_{\rm cr}$) and slow ($v \ll v_{\rm cr}$) motion of the mirror. Typical cavity response in case of slow and fast mirror motion is shown in Figure 2.

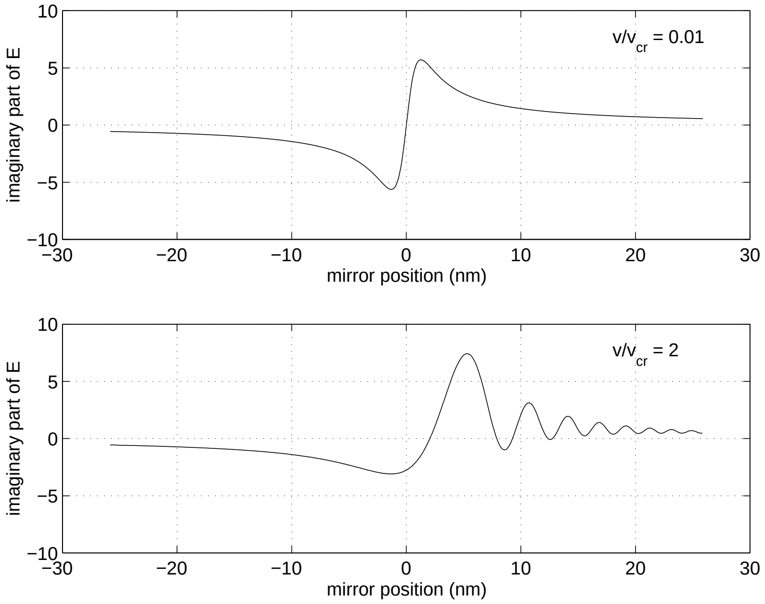

Figure 2: Modeled response of the LIGO 4km Fabry-Perot cavity (finesse 205.5). The upper plot corresponds to slow motion and the lower plot corresponds to fast motion of the mirror ($v_{\mathrm{cr}} = 1.48 \times 10^{-6}$ m/s).

The response of Fabry-Perot cavities is usually expressed in terms of amplitudes of the electro-magnetic field circulating in the cavity. The equation for dynamics of this field can be derived as follows. Let the amplitude of the incident wave be $E_{\mathrm{in}}(t)$ the amplitude of the wave inside the cavity be $E(t)$, both defined at the reflective surface of the input mirror as shown in Figure 1. Then the equation for the amplitude of the intra-cavity field is

$$E(t) = t_a E_{\mathrm{in}}(t) + r_a r_b e^{-2ikx(t-T)} E(t-2T), \tag{5}$$

where $t_a$ is the transmissivity of the input mirror and $k$ is the wave number ($k = 2\pi/\lambda$). During a short time when the mirror moves through the width of a resonance, which typically lasts milliseconds, the mirror velocity can be considered constant. Then the trajectory of the mirror can be approximated as linear:

$$x(t) = vt. \tag{6}$$

Equation (5) can be used for accurate simulation of the cavity field [6] for any given trajectory of its mirror. The cavity response to the steady motion of its mirror, shown in Figure 2, is generated using this equation.

Such numerical solutions of equation (5) provide an accurate description of the cavity field but give little insight into the physics of the process. Therefore, it is worthwile to obtain an approximate analytical solution for this equation.

An approximate solution can be found as follows. A general solution of equation (5) can be represented as a sum of two components:

$$E(t) = C(t) + D(t), \tag{7}$$

where $C(t)$ is a particular solution of the non-homogeneous equation and $D(t)$ is a general solution of the homogeneous equation:

$$\begin{aligned} C(t) - r_a r_b e^{-2ikv(t-T)} C(t-2T) &= t_a A, &(8)\\ D(t) - r_a r_b e^{-2ikv(t-T)} D(t-2T) &= 0. &(9) \end{aligned}$$

Here we assumed that the amplitude of the incident field is constant $E_{\rm in}(t) = A$. Both amplitudes, $C(t)$ and $D(t)$, change very little during one round-trip. In case of $C$-field the approximation $C(t-2T) \approx C(t)$ yields the solution:

$$C(t) \approx \frac{t_a A}{1 - r_a r_b e^{-2ikv(t-T)}}, \tag{10}$$

which is generally known as the adiabatic field. In case of $D$-field the approximation $D(t-2T) \approx D(t)$ yields only a trivial solution: $D(t) = 0$. Fortunately, the equation for $D$-field can be solved exactly. Let the initial value of the $D$-field be $D_0$. Then at any time, $t > 0$, its amplitude is given by

$$D(t) = D_0 \, (r_a r_b)^{t/2T} e^{i\phi(t)}, \tag{11}$$

where the phase of the $D$-field, $\phi(t)$, satisfies the equation

$$\phi(t) = \phi(t-2T) - 2kv(t-T). \tag{12}$$

Its solution is (up to an additive constant):

$$\phi(t) = -\frac{kv}{2T} t^2. \tag{13}$$

Thus, we obtain the solution for $D$-field:

$$D(t) = D_0 \, \exp\left[ -\frac{t}{\tau} - i\frac{kv}{2T} t^2 \right], \tag{14}$$

where $\tau$ is the cavity storage time, eq. (2). This expression is valid for $t > 0$ and describes the transient process due to the Doppler effect. The transient oscillates with the frequency

$$\Omega(t) \equiv \left|\frac{d\phi}{dt}\right| = \frac{k|v|}{T} t, \tag{15}$$

which linearly increases with time. The constant $D_0$ in eq.(14) can be found from the asymptotic behavior of the field [5] and is given by

$$D_0 = t_a A \sqrt{\frac{i\pi}{2kvT}} \exp\left[\frac{iT}{2kv\tau^2}\right]. \tag{16}$$

Note that the frequency of the oscillations, eq.(15), is equal to the accumulated Doppler shift of the frequency of light circulating in the cavity:

$$\Omega(t) = |\delta\omega| \frac{t}{2T}, \tag{17}$$

where $\delta\omega$ is the Doppler shift which occurs in one reflection off of a moving mirror, eq.(1).

Thus the transient which occurs during a free motion of the mirrors can be explained entirely by the Doppler effect. Based on the above equations we can briefly describe the generation of the transient as follows. As the mirror approaches a resonance position the light continuously builds up in the cavity. By the time the mirror passes a center of the resonance ($t = 0$) the cavity accumulates a substantial amount of light, which consists of the adiabatic and the Doppler components. Shortly after (typically after several round-trips) the light in the cavity becomes mostly dominated by the Doppler component. From then on the frequency of light circulating in the cavity is constantly shifting. As a result the light in the cavity becomes phase modulated and the complex amplitude begins oscillating at the frequency equal to the accumulated Doppler shift. At the same time the absolute value of the amplitude exponentially decreases with time; and the decay time is equal to the storage time of the cavity. This process is described by equation (14) for $t > 0$. To analyze the dynamics of the intra-cavity field for $t < 0$ numerical calculations based on equation (5) can be used.

The Doppler-induced oscillations of the intra-cavity field can be observed in the intensity of the transmitted light [3], which is proportional to

$$I(t) = |C(t) + D(t)|^2. \tag{18}$$

In this case the oscillations appear in the beats of $D$ and $C$ fields. (The absolute value of the amplitude of each field is not oscillating.) As a result

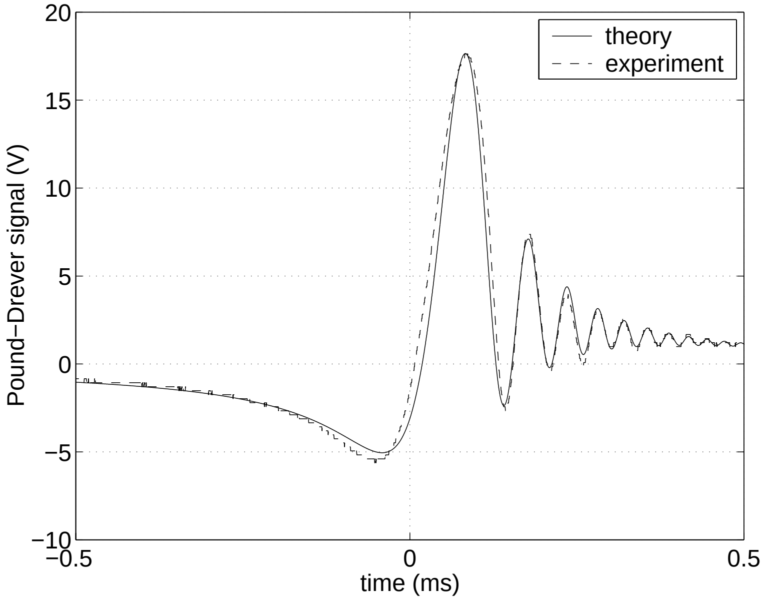

Figure 3: Transient response of the Fabry-Perot cavity of the Caltech 40m prototype ($v/v_{\rm cr} \approx 1.96$.)

variations of the intensity usually constitute a small fraction of the total intensity.

A direct way to observe the Doppler component is via the Pound-Drever signal which can be obtained by a coherent detection of the phase-modulated light reflected off the cavity [2]. Such a signal is approximately described by

$$V(t) = -{\rm Im}\{e^{i\alpha} E(t)\}, \tag{19}$$

where $\alpha$ is the phase of a local oscillator in the optical heterodyne detection. Since the Pound-Drever signal is proportional to the amplitude of the intracavity field the Doppler-induced oscillations are usually well observed in this method. Figure 3 shows the Pound-Drever signal of the 40m Fabry-Perot cavity recorded with a digital oscilloscope (dashed line) [7]. The theoretical prediction shown in the same Figure (solid line) was generated using equation (5). After adjustment of the demodulation phase ($\alpha \approx -0.09\pi$) the theoretical curve agrees reasonably well with the experimental curve.

The formula for the transient, eq.(14), can be used for extracting the cavity parameters from the Pound-Drever signal [7], [8]. In such analysis it is convenient to remove the adiabatic component from the Pound-Drever signal. The result can be approximately described by the function

$$V_D(t) = -|D_0| \exp\left[-\frac{t - t_0}{\tau}\right] \times \tag{20}$$

$$\sin\left[\alpha + \beta - \frac{kv}{2T}(t - t_0)^2\right], \tag{21}$$

where we introduced $t_0$ (the instance of mirror passing a center of the resonance) and $\beta = \arg D_0$. The Pound-Drever signal (with the adiabatic component removed) and the theoretical prediction based on the formula, eq. (20), is shown in Figure 4. By fitting the exponential decay, $\exp(-t/\tau)$, to the envelope of the oscillations, $|V_D(t)|$, we can find the storage time of the cavity and therefore its finesse. Applied to the data shown in Figure 4 this method yeilds the finesse of $1066 \pm 58$, which is close to the value known from the measurement of the mirror reflectivities (1050).

The oscillatory transient can also be used for measurements of the mirror velocity. This can be done by studying either peaks or zero-crossing of the signal $V_D(t)$. Consider the peaks of the signal. Let the time corresponding to the $n$th peak be $t_n$. An array of $t_n$ can be easily found from the measured trace of the signal $V_D(t)$. Theoretical values for $t_n$ are defined by the equation:

$$\cot\left[\alpha + \beta - \frac{kv}{2T}(t_n - t_0)^2\right] = -\frac{T}{\tau k v (t_n - t_0)}, \tag{22}$$

which can be solved perturbatively in terms of powers of $T/\tau$. Knowing the peak positions we can find the separations between the peaks $\Delta t_n$ and the positions of their midpoints $\bar{t}_n$ according to the equations

$$\Delta t_n = t_{n+1} - t_n, \tag{23}$$

$$\bar{t}_n = \frac{1}{2}(t_n + t_{n+1}). \tag{24}$$

The average frequency of oscillations of the Pound-Drever signal, defined as $\bar{\nu}_n = (2\Delta t_n)^{-1}$, can be found perturbatively in powers of $T/\tau$. In the lowest order they are defined by

$$\bar{\nu}_n = \frac{v}{\lambda T}(\bar{t}_n - t_0). \tag{25}$$

If the oscillations are close to being critically damped corrections to the linear dependence, eq.(25), become significant.

The first order correction is given by

$$\delta\bar{\nu}_n = -\frac{4\lambda v T(\bar{t}_n - t_0)^2}{\pi^2 \tau \left[16 v^2 (\bar{t}_n - t_0)^4 - \lambda^2 T^2\right]}. \tag{26}$$

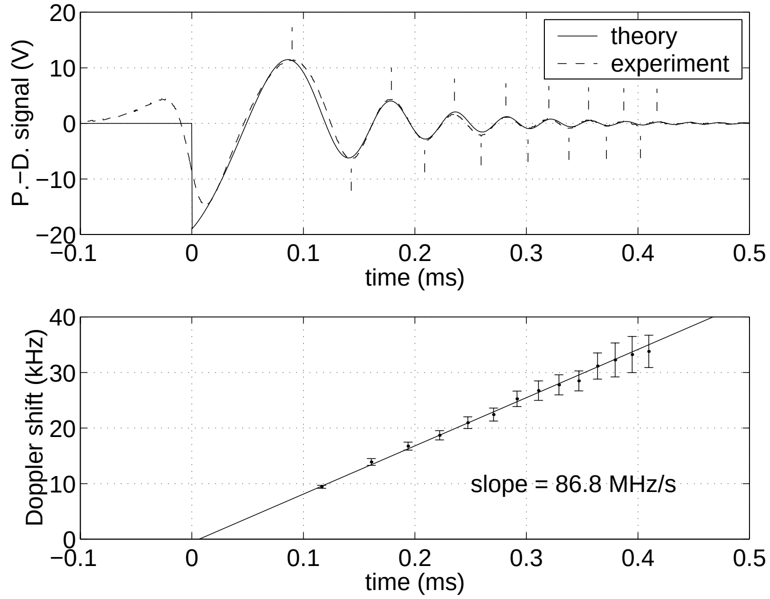

Figure 4: Upper plot shows the adjusted Pound-Drever signal, $V_D(t)$, for the 40m prototype. The theoretical prediction (solid line) and the experimental results (dashed line) matches well for $t > 0.02$ ms. The lower plot shows accumulation of the Doppler shift in time.

Since the oscillations shown in Figure 4 are far from being critically damped the first order correction can be neglected. As a result the measured values of the average frequencies fall very close to the linear dependence, eq.(25). Therefore, we can apply a linear fit to the data:

$$\bar{\nu}(t) = at + b. \tag{27}$$

The least square adjustment of the linear fit gives the slope and the intercept:

$$\begin{aligned} a &= (86.8 \pm 0.6) \times 10^6 \quad \text{Hz/s}, & (28) \\ b &= (-0.55 \pm 1.05) \times 10^3 \quad \text{Hz}. & (29) \end{aligned}$$

Note that the slope and the intercept are related to the mirror velocity ($v = \lambda T a$) and the instance of resonance ($t_0 = -b/a$). Therefore we obtain

$$\begin{aligned} v &= (5.74 \pm 0.40) \times 10^{-6} \quad \text{m/s}, & (30) \\ t_0 &= (0.64 \pm 1.2) \times 10^{-5} \quad \text{s}. & (31) \end{aligned}$$

The errors are due to uncertainty in the peak positions, which are limited in this measurement by the resolution of the oscilloscope. Further analysis of the Pound-Drever signal for Fabry-Perot cavity during lock acquisition can be found in [9].

In conclusion, the Doppler effect in Fabry-Perot cavities with suspended mirrors can be significant and reveals itself through the field transient, which can be directly observed via the Pound-Drever signal. The transient can be used for accurate measurements of the cavity finesse and the mirror velocities.

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