

Shot Noise in Gravitational-Wave Detectors with Fabry-Perot Arms

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The shot noise limited sensitivity is calculated for gravitational-wave interferometers with Fabry-Perot arms, similar to those being installed at the Laser Interferometer Gravitational-wave Observatory (LIGO) and the VIRGO facility. This calculation includes the effect of nonstationary shot noise due to phase modulation of the light. The resulting formula is experimentally verified using a test interferometer with suspended mirrors in the 40-meter arms.

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I Introduction

Interferometric gravitational wave detectors with multiple-kilometer baselines are currently under construction by the LIGO¹ project in the US and the VIRGO² project in Italy. Interferometers with baselines of several hundred meters are under construction in the GEO600³ projects in Germany and the TAMA⁴ project in Japan. These kilometer-scale detectors will be sensitive to relative displacements of their test masses of order 10^{-19} to 10^{-20} m/ $\sqrt{\text{Hz}}$ in the frequency band from approximately 10 Hz to several 1000 Hz. At frequencies above approximately 300 Hz the dominant noise source is expected to be photon shot noise.

The sensitivity limit imposed by photon shot noise depends on the optical configuration of the interferometer as well as the technique employed to read out the relative positions of the test masses. There has been considerable effort devoted to exploring novel optical configurations and readout schemes that improve the shot noise limited performance of the detectors for a given laser power without requiring unreasonably high power levels in the interferometer.^{5,6,7}

The optical configuration selected for the initial LIGO, VIRGO and TAMA detectors is a power recycled interferometer with Fabry-Perot arm cavities^{8,9} as shown in Figure 1. A passing gravitational wave incident from directly overhead will produce a fluctuating strain, that stretches one arm of the interferometer and contracts the other arm for half of the gravitational wave period. The interferometer measures the change in the difference between the arm lengths in a way directly analogous to a Michelson interferometer. The Fabry-Perot cavities are held on resonance by length control servos and the beam splitter is controlled so that the light returning from the two arms interferes destructively at the antisymmetric port. This light interferes constructively at the symmetric port, with light returning toward the laser in accordance with energy conservation. The very small deviation from resonance induced by a passing gravitational wave will cause the phase of the light reflected from the arms to change, spoiling the destructive interference at the antisym-

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metric port. This gives rise to the gravitational wave signal. The recycling mirror improves the shot noise limited sensitivity by redirecting the light returning to the laser back into the interferometer. The recycling mirror must be positioned so that the light it reflects back into the interferometer interferes constructively with the light transmitted through it from the laser.

Calculating the shot noise limited sensitivity of a gravitational wave interferometer is complicated by the fact that to achieve adequate sensitivity, the light in the interferometer is phase modulated. The output light power is time varying at the phase modulation frequency and its harmonics. Thus, the associated shot noise is nonstationary. Early treatments assumed that the shot noise was a white noise source with a variance proportional to the time-averaged power incident on the photo-detector.^{10,11,12} This approximation is useful for making order of magnitude predictions of shot noise limited sensitivity, but more accurate comparisons with experiment require including the effect of phase modulation and the demodulation waveform used. This has been done for a single Fabry-Perot cavity^{13,14} and for a delay line interferometer¹⁴ and the dependence of the shot noise on the demodulation waveform has been experimentally confirmed.^{15,16,17} A shot noise limited optical phase measurement has also been demonstrated at high optical power in a power recycled Michelson Interferometer.¹⁸

Fabry-Perot cavities are used in the arms of a power recycled Michelson interferometer to provide a large amplification of the optical phase shift generated by a gravitational wave. Here we give a detailed derivation of the shot noise limited displacement sensitivity for such a power-recycled interferometer with Fabry-Perot arm cavities which includes the effect of phase modulation applied to the light incident on the interferometer. The resulting formula is directly compared with data from a 40-m-long, suspended-mirror, test interferometer incorporating optical recombination of light returning from the two arms. The empirical method we develop for this comparison accurately determines the shot noise contribution to displacement noise even in the presence of other, larger noise contributions. The calculated shot noise limited displacement sensitivity and the experimental evaluation of this noise contribution are in good agreement within experimental uncertainties.

In Section II, we derive the response of the interferometer to mirror displacements. The power spectrum of shot noise at the demodulated signal output of the interferometer is given in Section III. Section IV describes an experimental confirmation of the calculated shot noise contribution to a test interferometer that uses Fabry-Perot arms with a 40-meter baseline.

II Interferometer Response

A recycled interferometer with the mirrors and fields labeled is shown in Figure 2. We specifically derive the contribution of shot noise in an interferometer that uses phase-modulation on the incident light^{19,20}. However the technique employed in this work is generally applicable to other configurations that apply phase modulation. The light incident from the laser is E_0 . The light is phase modulated, with modulation depth Γ , between the laser and the interferometer. This impresses sidebands on the light at frequencies above and below the laser frequency (carrier), separated by the modulation frequency and its harmonics. The carrier light leaving the antisymmetric port is E_A , which is typically very small in the absence of a signal because the antisymmetric port is held on a dark fringe for the carrier. Because of the asymmetry, the sidebands are not on a dark fringe at the antisymmetric port. We adopt the phase convention that the first order sidebands

have real amplitudes of opposite sign when incident on the beam splitter. The second order sidebands have equal real amplitudes of the same sign. We will neglect terms in the calculation of order Γ^3 because the modulation depth is assumed small. Thus, we will only need to consider up to second order sidebands. The transmission of the n th-order sideband from incidence on the beam splitter to the antisymmetric port is $\pm i \sin n\alpha$, where positive indicates the upper sideband and negative, the lower sideband²¹. The amplitude of the total complex field at the antisymmetric port is

$$E_{\text{anti}} = E_A + iE_+ \exp(i\omega t) + iE_+ \exp(-i\omega t) + iE_{2+} \exp(2i\omega t) - iE_{2+} \exp(-2i\omega t) \quad (1)$$

where ω is the angular modulation frequency, and E_+ and E_{2+} are the magnitudes of the first order and second order sideband fields at the antisymmetric port.

The detection system is modeled as a photodetector, a demodulator and a low-pass filter as shown in Figure 3. The low-pass filter need not be explicitly built as a separate element following the mixer output. In practice, all of the servo loops that derive their error signals from the mixer output have unity gain frequencies that are low compared to the modulation frequency. Thus, we shall ignore in our analysis any signals at the mixer output that are at or above the modulation frequency.

A gravitational wave interacting with the detector will produce the same differential mode signal as shaking mirror 4 by some other means. If we displace mirror 4 such that $x_4 = x_0 \sin \Omega t$, for sufficiently small x_0 this will produce a signal at the antisymmetric port given by:

$$E_A = E_{DC} - ik E_2 \frac{T_3 r_4}{(1 - r_3 r_4)^2} \frac{x_0 \sin(\Omega t + \psi)}{\sqrt{1 + (\Omega/\omega_c)^2}} \quad (2)$$

where k is the wave number of the light, E_{DC} is the field due to the contrast defect which comes from any non-interfering light on the photodetector and ψ is a phase factor irrelevant to this analysis. ω_c is the angular frequency of the so-called cavity pole,

$$\omega_c = \frac{c}{2l} \frac{1 - r_3 r_4}{r_3 r_4} \quad (3)$$

whose value is typically within the bandwidth of interest for gravitational waves. (These equations are derived in greater detail in Appendix A.) The modulation sidebands do not resonate in the arm cavities and thus are not affected by the motion of mirror 4.

Expressing the fields in units of $\sqrt{\text{photoelectrons/second}}$, simplifies the following formulae in our analysis. The photocurrent, i_p , is

$$\begin{aligned}
 i_p &= |E_A + iE_+ \exp(i\omega t) + iE_+ \exp(-i\omega t) + iE_{2+} \exp(2i\omega t) - iE_{2+} \exp(-2i\omega t)|^2 \\
 &= |E_A|^2 + 2E_+^2 + 4E_+ \text{Im}[E_A] \cos \omega t - 2E_+^2 \cos 2\omega t - 4E_{2+} \text{Re}[E_A] \sin 2\omega t
 \end{aligned} \tag{4}$$

The photocurrent has components at zero frequency (DC), ω and 2ω . The effect of the mixer and low pass filter is to pick out the ω component, which is

$$4kE_2E_+ \frac{T_3 r_4}{(1 - r_3 r_4)^2} \frac{x_0 \sin(\Omega t + \Psi)}{\sqrt{1 + (\Omega/\omega_c)^2}} \cos \omega t \tag{5}$$

Because the photocurrent is modulated, it is important to treat the shot noise as a nonstationary random process and to consider the actual demodulation waveform used.

The effective demodulation waveform used in the 40-m interferometer is cosinusoidal. Square wave demodulation is used at the mixer, but the band-pass filter, which is built into the photodiode and centered on the modulation frequency, makes this effectively cosinusoidal demodulation. This is because the square wave can be decomposed into a sum of cosine waves at odd multiples of the modulation frequency. Each cosine wave mixes with the corresponding component of the photocurrent to produce a signal after the low pass filter. The band-pass filter on the photodiode effectively eliminates all these higher frequency components in the photocurrent, so that only the fundamental cosine wave demodulation term is important.

Multiplying the component of the photocurrent at ω by $\cos \omega t$, we find

$$i_d = 4kE_2E_+ \frac{T_3 r_4}{(1 - r_3 r_4)^2} \frac{x_0 \sin(\Omega t + \Psi)}{\sqrt{1 + (\Omega/\omega_c)^2}} \cos^2 \omega t \tag{6}$$

The low-pass filter has a corner frequency that is much less than the modulation frequency. Thus, the component of $\cos^2 \omega t$ near DC will pass through, while the component at 2ω will not, so that

$$i_o = 2kE_2E_+ \frac{T_3 r_4}{(1 - r_3 r_4)^2} \frac{x_0 \sin(\Omega t + \Psi)}{\sqrt{1 + (\Omega/\omega_c)^2}} \tag{7}$$

We define $H(f)$ as the transfer function from x_0 to i_o .

$$\begin{aligned}
 |H(f)| &\equiv \left| \frac{\tilde{i}_o(f)}{\tilde{x}_0(f)} \right| \\
 &= 2k|E_2|E_+ \frac{T_3 r_4}{(1 - r_3 r_4)^2} \frac{1}{\sqrt{1 + (\Omega/\omega_c)^2}}
 \end{aligned} \tag{8}$$

where \tilde{i}_o and \tilde{x}_0 denote the Fourier transforms of i_o and x_0 .

III Noise

To quantify the noise performance of the interferometer, we must characterize the random process $\mathbf{x}(t)$ corresponding to the output in the absence of any signal. We shall use boldfaced symbols in our notation here to mean random processes and $E\{\}$ to mean the expectation value or ensemble average. Early treatments of the shot noise assumed it was stationary and ignored the effect of the modulation of the photocurrent. Stationary noise is most conveniently represented using the one-sided power spectrum $S_{xx}(f)$ of $\mathbf{x}(t)$. $S_{xx}(f)$ is defined as the Fourier transform of the autocorrelation function $R_{xx}(\tau)$ of $\mathbf{x}(t)$:

$$\begin{aligned} R_{xx}(\tau) &= E\{\mathbf{x}(t + \tau)\mathbf{x}(t)\} \\ S_{xx}(f) &= 2 \int_{-\infty}^{\infty} R_{xx}(\tau) \exp(2\pi if\tau) d\tau \end{aligned} \quad (9)$$

If $\mathbf{x}(t)$ is the input of a linear system, whose transfer function is $H(f)$, and $\mathbf{y}(t)$ is the output, then

$$S_{yy}(f) = |H(f)|^2 S_{xx}(f) \quad (10)$$

The output $\mathbf{i}_o(t)$ of our model, in the absence of a signal, is not stationary since it fluctuates at the modulation frequency. However, it is cyclostationary, which is to say that for any t , the statistics of $\mathbf{i}_o(t)$ are the same as those of $\mathbf{i}_o(t+T)$, where T is the period of modulation. In this situation if we define the average autocorrelation and power spectrum:

$$\begin{aligned} \overline{R}_{xx}(\tau) &= \frac{1}{T} \int_t^{t+T} R_{xx}(t' + \tau, t') dt' \\ \overline{S}_{xx}(f) &= 2 \int_{-\infty}^{\infty} \overline{R}_{xx}(\tau) \exp(2\pi if\tau) d\tau \end{aligned} \quad (11)$$

Then the relation

$$\overline{S}_{yy}(f) = |H(f)|^2 \overline{S}_{xx}(f) \quad (12)$$

holds true. Averaging in this way is equivalent to modeling the time reference or phase of the cyclostationary process as a random variable that is uniformly distributed over one cycle. In this case the phase-randomized process is stationary.^{22,23}

Our goal then is to calculate $\overline{S_{i_o i_o}}(f)$, the average power spectrum of the interferometer output. We begin by finding $\overline{S_{i_d i_d}}(f)$. The details of the derivation are in Appendix B; the result is:

$$\begin{aligned} \overline{S_{i_d i_d}}(f) = & 3E_+^2 + E_{DC}^2 \\ & + (9E_+^4 + 6E_{DC}^2 E_+^2 + E_{DC}^2 + 4E_{2+}^2 E_{DC}^2) \delta(2\pi f - \omega) \\ & + (E_+^4 + 4E_{2+}^2 E_{DC}^2) \delta(2\pi f - 3\omega) \end{aligned} \quad (13)$$

This power spectrum has two sharp components, one at the modulation frequency, and one at its third harmonic, as well as a broadband component. Only the broadband component interests us, since it falls into the gravitational wave frequency band. The low-pass filter in our model of the detection system will leave this part of the noise spectrum unaffected and will attenuate the very high frequency components. Therefore,

$$\overline{S_{i_o i_o}}(f) = 3E_+^2 + E_{DC}^2 \quad (14)$$

Finally, the displacement noise in one test mass equivalent to shot noise is found by substituting from Eq. (8) and Eq. (14):

$$\begin{aligned} \overline{S_{x_4 x_4}}(f)^{1/2} &= \frac{\overline{S_{i_o i_o}}(f)^{1/2}}{|H(f)|} \\ &= \frac{\sqrt{3E_+^2 + E_{DC}^2}}{2k|E_2|E_+} \frac{(1 - r_3 r_4)^2}{T_3 r_4} \sqrt{1 + \left(\frac{2\pi f}{\omega_c}\right)^2} \end{aligned} \quad (15)$$

IV Experiment

We have derived the differential mode displacement equivalent to shot noise in a power recycled interferometer with Fabry-Perot arm cavities. The 40-m interferometer on the Caltech campus provides us with an opportunity to compare the theory to measurement. From April, 1995 to August, 1996 the 40-m interferometer was operated in a recombined configuration, which is identical to the planned initial LIGO and VIRGO configurations without the recycling mirror.²⁴ A recombined interferometer can be treated as a power recycled interferometer with recycling mirror transmission equal to one.

To compare our theoretical expression for shot noise with laboratory measurements we must determine the reflectivities and transmissions of the arm cavity mirrors as well as the fields present in the interferometer. The transmissions and losses of the mirrors can be obtained from in-situ measurements using the “ringdown” technique.²⁵ This technique consists of building up a resonant field inside the cavity and then shutting off the power incident on the cavity. Observing the time scale of the exponential decay of the light leaking out of the cavity allows a calculation of the mirror parameters.²⁶ The measured parameters are shown in Table 1.

The fields in the interferometer, however, are not available for direct measurement. What we instead measure is the DC voltage obtained by passing the antisymmetric port photocurrent through a known resistor. We record the minimum voltage when the interferometer is in lock (V_{\min}) and the maximum voltage observed when the arm cavities are out of lock and the beam splitter is allowed to swing freely (V_{\max}). The modulation depth (Γ) is measured using an optical spectrum analyzer. The fields are then found from

$$E_2 = \sqrt{\frac{V_{\max}}{R e}} J_0(\Gamma) \quad (16)$$

$$E_+ = \sqrt{\frac{V_{\max}}{R e}} J_1(\Gamma) \sin \alpha \quad (17)$$

$$E_{DC} = \sqrt{\frac{V_{\min}}{R e} - 2E_+^2} \quad (18)$$

where R is the resistance in series with the photodiode and e is the charge of the electron in Coulombs. (Note that for comparison with experiment, we continue to write the fields in units of $\sqrt{\text{photoelectrons/second}}$ as we have done for the theoretical expressions.)

To include the effect of light that is not mode matched properly into the arm cavities, we also measure the mode-matching fraction, M .

$$M \approx \frac{1 - R_{\text{arm}}}{1 - R_{\text{theory}}} \quad (19)$$

where R_{arm} is the reflectivity of the arm cavities on resonance and R_{theory} is the theoretical reflectivity for a perfectly aligned cavity with the same mirror transmissions and losses. The mode-matching fraction affects the shot noise limit because only the light that could mode match into the cavities produces the signal. Mode matching does not affect the noise except as already accounted for in E_{DC} . Thus the effective magnitude of E_2 and E_+ in the denominator of the shot noise expression (Eq. (15)) are reduced by \sqrt{M} . So,

$$\overline{S}_{\Delta}(f)^{1/2} = \frac{\sqrt{3E_+^2 + E_{DC}^2}}{2kM|E_2|E_+} \frac{(1 - r_3 r_4)^2}{T_3 r_4} \sqrt{1 + \left(\frac{2\pi f}{\omega_c}\right)^2} \quad (20)$$

The parameters used in the shot noise calculation are collected in Table 2. The resulting curve is shown in Figure 4.

We would like to compare this calculated curve to an empirical measurement of the shot noise contribution to the gravitational wave signal (discussed below) and to the interferometer displacement spectrum taken at the time these measurements were done. The interferometer displacement spectrum can be obtained by monitoring a test point in the servo system electronics used to control differences in the lengths of the Fabry-Perot cavities when the interferometer is held on resonance with a dark fringe at the antisymmetric port. This signal can then be calibrated by actuating a mirror (mirror 4 of Figure 2) to produce known sinusoidal displacements at a frequency which is swept through the frequency range of interest. An empirical measurement of the shot noise contribution to the gravitational wave signal can be made by blocking the laser light and shining incandescent light on the antisymmetric photodiode such that the photocurrent is the same as in normal operation. The gravitational wave readout equivalent to this shot noise can then be calibrated, provided the effect of the loop gain of servo systems controlling the interferometer is properly taken into account. With the interferometer in lock, the shot noise signal is suppressed by the differential mode loop gain. When the laser light is blocked, the differential mode loop is open and this suppression factor is no longer present.

The action of changing loop gain on various noise sources can be illustrated by a simple loop analysis. The differential mode servo loop, with the places where shot noise, dark noise of the photodiode and readout noise would sum in, is shown in Figure 5. The transfer functions from the noise inputs to the gravitational wave readout in the open loop case (when the laser light is blocked) are:

$$\left. \frac{x}{s} \right|_{\text{open loop}} = ABC \quad \left. \frac{x}{n} \right|_{\text{open loop}} = C \quad (21)$$

With the loop closed during normal interferometer operation,

$$\left. \frac{x}{s} \right|_{\text{closed loop}} = \frac{ABC}{1 - L} \quad \left. \frac{x}{n} \right|_{\text{closed loop}} = C \quad (22)$$

where the open loop gain is $L = ABP$. Thus with the loop closed, shot noise and the dark noise of the photodiode are suppressed by $1/(1 - L)$ relative to the open loop measurement while the readout noise is unaffected.

The complete measurement procedure for the empirical measurement of the shot noise limit shown in Figure 4 follows. The transfer functions of the differential mode servo loop are measured to obtain the loop correction factor $1/(1 - L)$. After taking an interferometer displacement spectrum and the transfer function necessary for calibration, the laser light is blocked. As a check of the readout noise, the input to the readout electronics is terminated in 50Ω and the power spec-

trum of the gravitational wave readout is recorded. After reconnecting the readout electronics, the power spectrum of the gravitational wave readout is recorded with no light on the antisymmetric photodiode. This is the dark noise spectrum and should be well above the level due to noise in the readout electronics, as it was in every case. Finally, the photodiode was illuminated with incandescent light to achieve the same photocurrent as is present during normal interferometer operation. The resulting power spectrum is the shot plus dark noise spectrum. The power spectrum of shot noise alone is recovered by quadrature subtraction of the dark noise. The shot noise power spectrum is then increased by 1 dB to reflect the fact that the measured fluctuations in the photocurrent from the photodiode are observed to be 1 dB greater for the green laser light than for incandescent light producing the same DC photocurrent. (The origin of this effect is not understood²⁷.) This spectrum is then divided by $1/(1-L)$, to account for the differential mode loop gain, and calibrated as usual to convert it into an equivalent amount of displacement noise.

The resulting empirical measurement of the shot noise equivalent displacement is shown as the dotted line in Figure 4. Ideally, the shot noise power spectrum should have been larger than the dark noise spectrum by a reasonable margin. In fact, for the measurement shown in the figure, there was only approximately a 3 dB margin which is why the resulting estimate for the shot noise contribution alone appears noisy.

It is evident from Figure 4 that the measured contribution of shot noise to the interferometer output is less than the total noise. A significant amount of effort was made to understand the observed excess noise in the interferometer displacement spectrum. It is suggestive that the shape of the spectrum matches that predicted for shot noise above approximately 600 Hz. This would be the case for any noise source that is equivalent to white noise at the demodulator output. A number of potential noise sources were explicitly tested for, including intensity noise, frequency noise, beam splitter motion and shot noise in the auxiliary signals. None of these noise sources were found to limit the interferometer displacement spectrum above 600 Hz.

Intensity noise can contribute to the displacement spectrum due to in-band ($f < 10\text{kHz}$) fluctuations as well as due to fluctuations at frequencies near the radio frequency (RF) modulation frequency. The in-band contribution was estimated by injecting white intensity noise at a level to clearly show up in the interferometer output above the observed noise level. This drive level was then doubled to check for linearity, which did produce a 6 dB increase in the interferometer noise level. By comparing the increase in the interferometer displacement spectrum to the increase in the intensity noise spectrum, a limit on the intensity noise contribution could be set. The in-band intensity noise contribution was $3 \times 10^{-19} \text{ m}/\sqrt{\text{Hz}}$ at 850 Hz and was relatively flat from 500 - 1000 Hz. By contrast, the interferometer noise was $(1 - 2) \times 10^{-18} \text{ m}/\sqrt{\text{Hz}}$ over this frequency range. A test for RF intensity noise is to misalign all the test masses except for a single vertex mass so that light incident on the interferometer is reflected back to the photodiodes. After measuring the demodulated signal at the symmetric or antisymmetric photodiodes, the laser light is blocked and the same amount of power is applied to the photodiode with an incandescent light source. Above 200 Hz the spectra of the resulting demodulated signal was identical in both cases. This confirms that the intensity noise of the light is shot noise limited near the RF modulation frequency.

The frequency noise contribution to the interferometer output was estimated by injecting a monochromatic frequency deviation and observing the resulting peaks in the frequency control servo signal and in the interferometer output. We expect the frequency noise feeding through to

the interferometer output to be essentially constant over some small region around the injected peak. By comparing the peak-to-background measurements, the estimated frequency noise contribution to the interferometer output at 750 Hz was determined to be less than $7 \times 10^{-20} \text{ m}/\sqrt{\text{Hz}}$.

The contribution from in-band fluctuations in the beam splitter position was estimated by measuring the transfer function between the beam splitter feedback signal and the interferometer output. The ambient spectrum of the beam splitter feedback was multiplied by this transfer function to find the estimate. Above 600 Hz, the contribution to the interferometer output from beam splitter motion is more than 40 dB below the observed spectrum.

Shot noise in the auxiliary servo signals may feed through onto the gravitational readout signal. The error signals for these servos are measured at the symmetric photodiode. We placed an attenuator before the symmetric photodiode to halve the laser light and then used an incandescent light source to increase the power on the photodiode by a factor of four. We saw no observable change in the gravitational wave spectrum.

V Conclusion

We have given a derivation of the shot noise limited sensitivity of a power recycled interferometer with Fabry-Perot arm cavities. The result was compared with data from the 40-m interferometer operated in a recombined configuration without a recycling mirror. In particular an empirical measurement of the contribution of shot noise to the interferometer was possible, even in the presence of other noise sources. This empirical measurement of the shot noise contribution agrees with the calculation to within the uncertainties of the parameters in the calculation and in the calibration, typically a few dB.

We determined that the interferometer was not limited by shot noise at any frequency. Over the frequency range from 500-1500 Hz, the interferometer exhibited a noise equivalent displacement that was typically $(1 - 2) \times 10^{-18} \text{ m}/\sqrt{\text{Hz}}$ (except for narrow features associated with mechanical resonances and line harmonics), increasing to approximately $7 \times 10^{-18} \text{ m}/\sqrt{\text{Hz}}$ at 5000 Hz. The measured contribution of shot noise to the noise equivalent displacement varied from $3 \times 10^{-19} \text{ m}/\sqrt{\text{Hz}}$ to approximately $3 \times 10^{-18} \text{ m}/\sqrt{\text{Hz}}$ over the frequency range 500-5000 Hz. The calculated shot noise equivalent displacement, using measured parameters in Table 2, was larger than the measured displacement by approximately 3 dB. This is comparable to our estimate of measurement uncertainties.

We confirmed that shot noise was not the dominant noise by attenuating the light leaving the antisymmetric port by 37.5% and directing light from an incandescent bulb onto the photodiode to raise the incident power by a factor of 3.2. We would expect a 7 dB increase in the interferometer displacement spectrum if it were limited by shot noise, but the largest increase seen anywhere in this frequency band was 4 dB. Although the interferometer noise is not fully understood it is clearly not shot noise limited. A number of noise sources were explored and eliminated as significant noise contributions, including laser intensity and frequency fluctuations, beam splitter motion and shot noise on the auxiliary control signals derived from the symmetric port. A leading candidate to explain the excess noise is scattered light, most likely in the vertex area. There was significant scattering from optics situated inside the beam splitter's vacuum chamber and we were not able to extensively test whether this caused the noise excess. However the presence of this

noise did not degrade our ability to confirm the shot noise contribution to the observed displacement spectrum as shown in Figure 4.

The methods used here for calculation of the shot noise contribution and for the empirical measurement of this contribution are quite general. They are directly applicable to the large-scale gravitational-wave detectors currently under construction for LIGO and VIRGO, and they can be readily adapted for other interferometer configurations.

Appendix A: Effect of Shaking an End Mirror

Here we derive the effect on E_A of shaking mirror 4 at frequency Ω as mentioned in Section II. To do this we will need to calculate the field reflected from the arm cavity, E_5 , in terms of the incident field, E_4 . This is done in two steps. First, we solve for E_5 given a small DC displacement of mirror 4. Then we generalize this result to frequencies in the gravitational wave band.

Consider a small displacement of mirror 4 away from the carrier resonance. Let $x_4 = 0$ on resonance so that $x_4 = x_0$ after the displacement. We define the arm cavity reflectivity away from resonance to be $r_{arm}(\phi)$ such that $E_5 = r_{arm}(\phi) E_4$, where

$$r_{arm}(\phi) = \frac{r_3 - (1 - L_3) r_4 \exp(i\phi)}{1 - r_3 r_4 \exp(i\phi)} \quad \text{A-1}$$

$$\phi = 2 k x_4$$

Here L_3 is the loss associated with mirror 3 (assumed equal to the loss from mirror 4). Taylor expand $r_{arm}(\phi)$:

$$E_5 = E_4 \left(r_{arm}|_{x_4=0} + \left. \frac{dr_{arm}}{dx_4} \right|_{x_4=0} x_0 + \dots \right) \quad \text{A-2}$$

Taking the derivative and noting $d\phi/dx_4 = 2k$, for sufficiently small x_0 we can write

$$E_5 = E_4 \left[\frac{r_3 - (1 - L_3) r_4}{1 - r_3 r_4} - 2ik \frac{T_3 r_4}{(1 - r_3 r_4)^2} x_0 \right] \quad \text{A-3}$$

Now let $x_4 = x_0 \sin \Omega t$. Note that in the small amplitude limit we are considering, shaking the rear mirror at frequency Ω phase modulates the light reflected from the mirror. This impresses sidebands on the reflected light at frequencies Ω above and below the carrier frequency. The transmission of these sidebands from the rear mirror through the cavity is,

$$t_{arm}(\phi) = \frac{t_3 \exp(i\phi/2)}{1 - r_3 r_4 \exp(i\phi)} \quad \text{A-4}$$

Now, $\phi = 2\omega l/c$ where ω is the angular frequency of the light and l is the length of the cavity. The frequency of the light with the impressed sidebands from the mirror motion is $\omega = \omega_0 \pm \Omega$ where ω_0 is the carrier resonance frequency. We assume Ω is small compared to the cavity free spectral range. (The arm cavities for LIGO will have a free spectral range of 37.5 kHz and the free spectral range for the 40-m interferometer was 3.75 MHz.) We can approximate:

$$\exp[i\phi] = \exp\left[2i(\omega_0 \pm \Omega)\frac{l}{c}\right] = \exp\left[i\Omega\frac{2l}{c}\right] \approx 1 \pm i\Omega\frac{2l}{c} \quad \text{A-5}$$

$$t_{arm}(\phi) = \frac{t_3\left(1 \pm i\Omega\frac{l}{c}\right)}{1 - r_3 r_4\left(1 \pm i\Omega\frac{2l}{c}\right)} \quad \text{A-6}$$

This has a zero at angular frequency c/l , which is twice the cavity free spectral range. This zero is well above the gravitational wave band and therefore not of interest. There is also a pole at angular frequency

$$\omega_c = \frac{c}{2l} \frac{1 - r_3 r_4}{r_3 r_4} \quad \text{A-7}$$

This is the so-called *cavity pole*. It will typically be important and lie in the gravitational wave band.

We can now generalize from the DC case by noting that all the frequency dependence in the transfer function from x_4 to E_5 is contained in a single pole.

$$E_5 = E_4 \left[\frac{r_3 - (1 - L_3) r_4}{1 - r_3 r_4} - 2ik \frac{T_3 r_4}{(1 - r_3 r_4)^2} \frac{x_0 \sin(\Omega t + \psi)}{\sqrt{1 + (\Omega/\omega_c)^2}} \right] \quad \text{A-8}$$

where ψ is a phase factor which is irrelevant for this analysis.

If we assume negligible losses and equal power transmission and reflection in the beam splitter,

$$E_4 = \frac{E_2}{\sqrt{2}} \quad E_6 = -\frac{E_2}{\sqrt{2}} \quad \text{A-9}$$

The field at the antisymmetric port is

$$E_A = \frac{1}{\sqrt{2}} E_5 + \frac{1}{\sqrt{2}} E_7 \quad \text{A-10}$$

Now,

$$E_7 = E_6 \frac{r_5 - (1 - L_5) r_6}{1 - r_5 r_6} \quad \text{A-11}$$

and

$$E_A = E_{DC} - i k E_2 \frac{T_3 r_4}{(1 - r_3 r_4)^2} \frac{x_0 \sin(\Omega t + \psi)}{\sqrt{1 + (\Omega/\omega_c)^2}} \quad \text{A-12}$$

where E_{DC} is the excess light at the antisymmetric port due to the imperfect matching of mirror parameters between the two arms or, more generally, any non-interfering sources of light on the photodetector.

Appendix B: Average Power Spectrum of the Demodulator Output

In this appendix, we derive the average power spectrum of the demodulator output, $\overline{S_{i_{d,d}}}(f)$, from the time-averaged autocorrelation function, $\overline{R_{i_{d,d}}}(\tau)$, using the methods discussed in reference 21. To calculate $\overline{R_{i_{d,d}}}(\tau)$, we first find the expectation value of the photocurrent, $E\{\mathbf{i}_p(t)\}$, in the absence of any signal. In this case $E_A = E_{DC}$, thus from Eq. (4):

$$\begin{aligned} E\{\mathbf{i}_p(t)\} &= |E_{DC}|^2 + 2E_+^2 + 4E_+ \text{Im}[E_{DC}] \cos \omega t \\ &\quad - 2E_+^2 \cos 2\omega t - 4E_{2+} \text{Re}[E_{DC}] \sin 2\omega t \\ &= E_{DC}^2 + 2E_+^2 - 2E_+^2 \cos 2\omega t - 4E_{2+} E_{DC} \sin 2\omega t \end{aligned} \quad \text{B-1}$$

E_{DC} has no imaginary part, because if it had, the length control servo would induce a differential change in the cavity lengths to cancel it.

The total number of electrons having left the photodetector since some initial time $t = 0$ is modeled as a non-uniform Poisson process. A Poisson process $q(t)$ is a random process which is constant except for unit increments at random points in time, t_i . We label $\lambda(t)$ the density of the points of t_i . The term *non-uniform* applies if the density of points is a function of time. We identify $\lambda(t) = E\{\mathbf{i}_p(t)\}$. We write this as

$$\lambda(t) = a + b \cos 2\omega t + c \sin 2\omega t \quad \text{B-2}$$

The photodetector output current is then a random process that is the derivative of a Poisson process. This is called a process of Poisson impulses.

$$\begin{aligned} \mathbf{i}_p(t) &= \frac{d\mathbf{q}(t)}{dt} \\ &= \sum_i \delta(t - t_i) \end{aligned} \tag{B-3}$$

The autocorrelation of a non-uniform Poisson process is:²⁸

$$R_{qq}(t_1, t_2) = \begin{cases} \int_0^{t_2} \lambda(t) dt \left[1 + \int_0^{t_1} \lambda(t) dt \right] & t_1 > t_2 \\ \int_0^{t_1} \lambda(t) dt \left[1 + \int_0^{t_2} \lambda(t) dt \right] & t_2 > t_1 \end{cases} \tag{B-4}$$

The autocorrelation of the derivative of a random process is given by²⁹

$$R_{q'q'}(t_1, t_2) = \frac{\partial^2 R_{xx}(t_1, t_2)}{\partial t_1 \partial t_2} \tag{B-5}$$

Since $\mathbf{i}_p(t) = \mathbf{x}'(t)$ we have only to substitute into the equation above to find the autocorrelation for the photocurrent. So,

$$\begin{aligned} R_{i_p i_p}(t_1, t_2) &= \frac{\partial^2 R_{qq}(t_1, t_2)}{\partial t_1 \partial t_2} \\ &= \begin{cases} \lambda(t_1) \lambda(t_2) & t_1 > t_2 \\ \lambda(t_2) \lambda(t_1) & t_2 > t_1 \end{cases} \end{aligned} \tag{B-6}$$

However, there is a discontinuity in the derivative at $t_1 = t_2$. Thus,

$$R_{i_p i_p}(t_1, t_2) = \lambda(t_1) \lambda(t_2) + \lambda(t_1) \delta(t_1 - t_2) \tag{B-7}$$

We can use this result to find the time-averaged autocorrelation of the demodulator output $\mathbf{i}_d(t) = \mathbf{i}_p(t) \cos \omega t$:

$$\begin{aligned} R_{i_d i_d}(t + \tau, t) &= E\{\mathbf{i}_p(t + \tau) \cos \omega(t + \tau) \mathbf{i}_p(t) \cos \omega t\} \\ &= E\{\mathbf{i}_p(t + \tau) \mathbf{i}_p(t)\} \cos \omega(t + \tau) \cos \omega t \\ &= [\lambda(t + \tau) \lambda(t) + \lambda(t + \tau) \delta(\tau)] \cos \omega(t + \tau) \cos \omega t \end{aligned} \tag{B-8}$$

$$\begin{aligned}
 \overline{R_{i_d i_d}}(\tau) &= \frac{1}{T} \int_0^T R_{i_d i_d}(t + \tau, t) dt \\
 &= \frac{1}{T} \int_0^T [\lambda(t + \tau) \lambda(t) + \lambda(t + \tau) \delta(\tau)] \cos \omega(t + \tau) \cos \omega t dt
 \end{aligned}
 \tag{B-9}$$

where T is the modulation period.

To find the average power spectrum, we take the Fourier transform of the average autocorrelation:

$$\overline{S_{i_d i_d}}(f) = 2 \int_{-\infty}^{\infty} \overline{R_{i_d i_d}}(\tau) \exp(2\pi i f \tau) d\tau
 \tag{B-10}$$

We will evaluate the two terms in Eq. B-9 one at a time. The first term gives

$$\begin{aligned}
 &\frac{1}{T} \int_0^T \lambda(t + \tau) \lambda(t) \cos \omega(t + \tau) \cos \omega t dt \\
 &= \frac{1}{T} \int_0^T [a + b \cos 2\omega t + c \sin 2\omega t] \\
 &\quad [a + b \cos 2\omega(t + \tau) + c \sin 2\omega(t + \tau)] \cos \omega(t + \tau) \cos \omega t dt \\
 &= \frac{1}{2} \left[\left(a^2 + ab + \frac{1}{4}(b^2 + c^2) \right) \cos \omega \tau + \frac{1}{4}(b^2 + c^2) \cos 3\omega \tau \right]
 \end{aligned}
 \tag{B-11}$$

Substituting this into Eq. B-10 yields

$$\begin{aligned}
 &2 \int_{-\infty}^{\infty} \frac{1}{2} \left[\left(a^2 - 2ab + \frac{1}{4}(b^2 + c^2) \right) \cos \omega \tau + \frac{1}{4}(b^2 + c^2) \cos 3\omega \tau \right] \exp(2\pi i f \tau) d\tau \\
 &= \left(a^2 + ab + \frac{1}{4}(b^2 + c^2) \right) \delta(2\pi f - \omega) + \frac{1}{4}(b^2 + c^2) \delta(2\pi f - 3\omega)
 \end{aligned}
 \tag{B-12}$$

To evaluate the second term we will reverse the order of integration:

$$\begin{aligned}
& \frac{2}{T} \int_{-\infty}^{\infty} \int_0^T \lambda(t+\tau) \delta(\tau) \cos \omega(t+\tau) \cos \omega t \, dt \exp(2\pi i f \tau) \, d\tau \\
&= \frac{2}{T} \int_0^T \int_{-\infty}^{\infty} \lambda(t+\tau) \delta(\tau) \cos \omega(t+\tau) \cos \omega t \exp(2\pi i f \tau) \, d\tau \, dt \\
&= \frac{2}{T} \int_0^T \lambda(t) \cos^2 \omega t \, dt \\
&= \frac{2}{T} \int_0^T (a + b \cos 2\omega t + c \sin 2\omega t) \cos^2 \omega t \, dt \\
&= a + \frac{b}{2}
\end{aligned} \tag{B-13}$$

Therefore, the average power spectrum of the demodulator output is

$$\begin{aligned}
\overline{S_{i_d i_d}}(f) &= 3E_+^2 + E_{DC}^2 \\
&+ (9E_+^4 + 6E_{DC}^2 E_+^2 + E_{DC}^2 + 4E_{2+}^2 E_{DC}^2) \delta(2\pi f - \omega) \\
&+ (E_+^4 + 4E_{2+}^2 E_{DC}^2) \delta(2\pi f - 3\omega)
\end{aligned} \tag{B-14}$$

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- 29.Papoulis, *ibid*, p. 313-314, equation (10-95).

Table 1: Parameters for the 40-m Interferometer

Parameter	Symbol	Value
Mirror (power) transmissions	T_2	0.45
	T_3	280 ppm
	T_5	300 ppm
	T_4, T_6	12 ppm
Loss in each mirror	L_3, L_4	110 ppm
	L_5, L_6	56 ppm
Asymmetry	δ	50.8 cm
Modulation frequency	f_{mod}	12.33 MHz
Modulation index	Γ	1.49
Contrast defect	$1 - C$	0.03

Table 2: Parameters Used in Shot Noise Calculation

Table 2:	
Name	Value
V_{\max}	1.1 V
V_{\min}	20 mV
R	50 Ω
Γ	0.705
M	0.77
α	0.132
T_3	280 ppm
r_3	0.999805
r_4	0.999938

Figure Captions:

Figure 1: Power recycled interferometer with Fabry-Perot arm cavities.

Figure 2: Recycled interferometer with mirrors and optical fields labelled.

Figure 3: Detection system for the antisymmetric port light.

Figure 4: Calculated shot noise contribution to interferometer displacement spectrum (smooth, long-dashed curve), with empirical measurement of shot noise contribution (short-dashed) and interferometer displacement spectrum taken shortly before on January 10, 1996 (solid).

Figure 5: Differential mode servo loop with shot noise, dark noise and readout noise inputs.

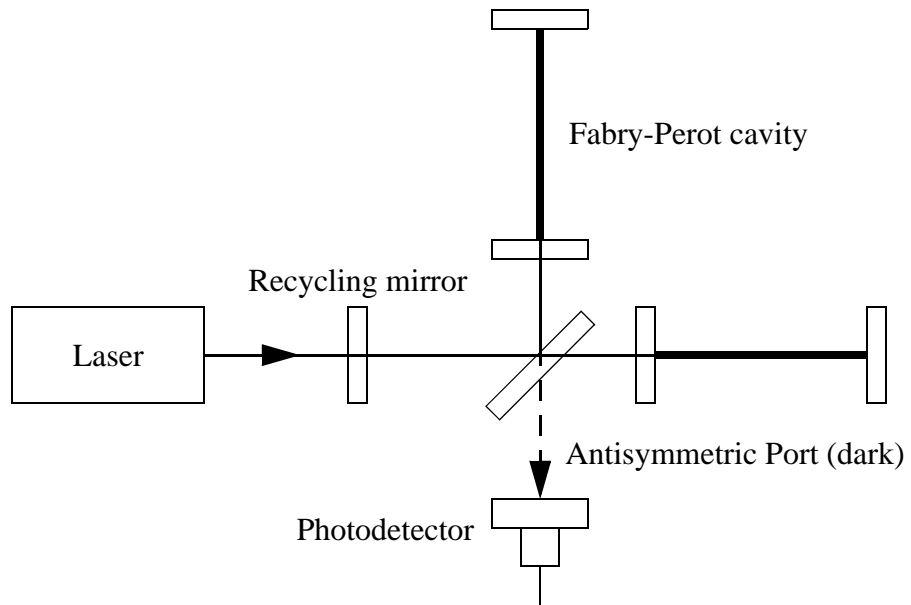


Figure 1

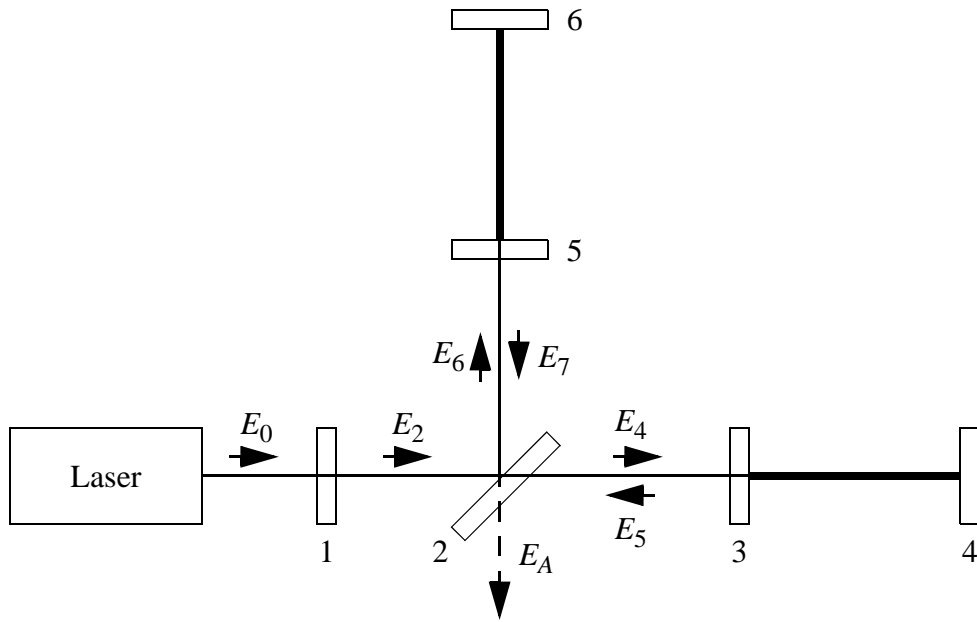


Figure 2

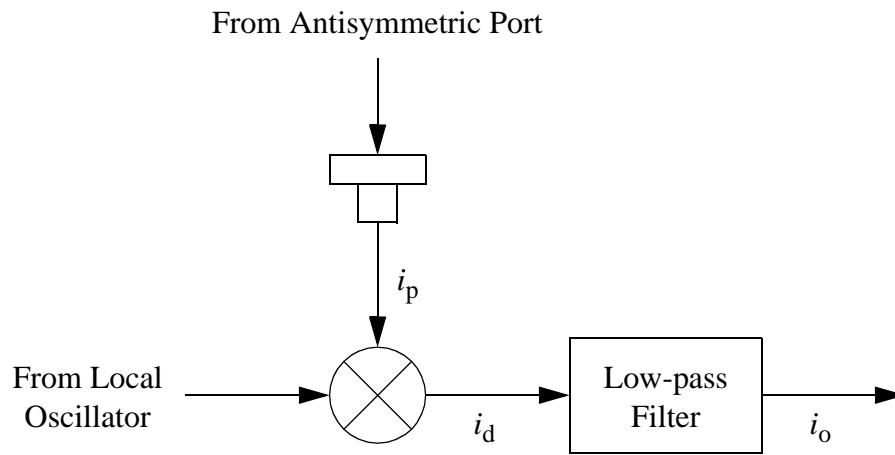


Figure 3

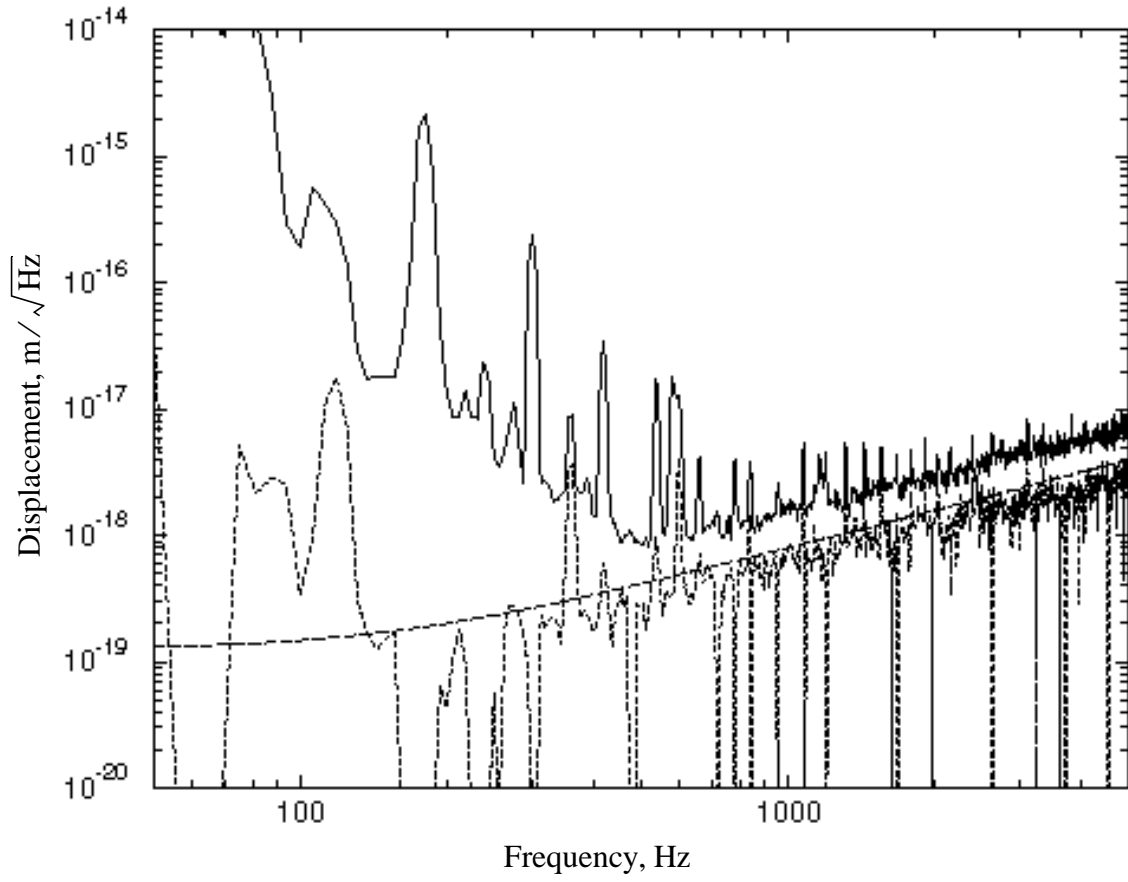


Figure 4

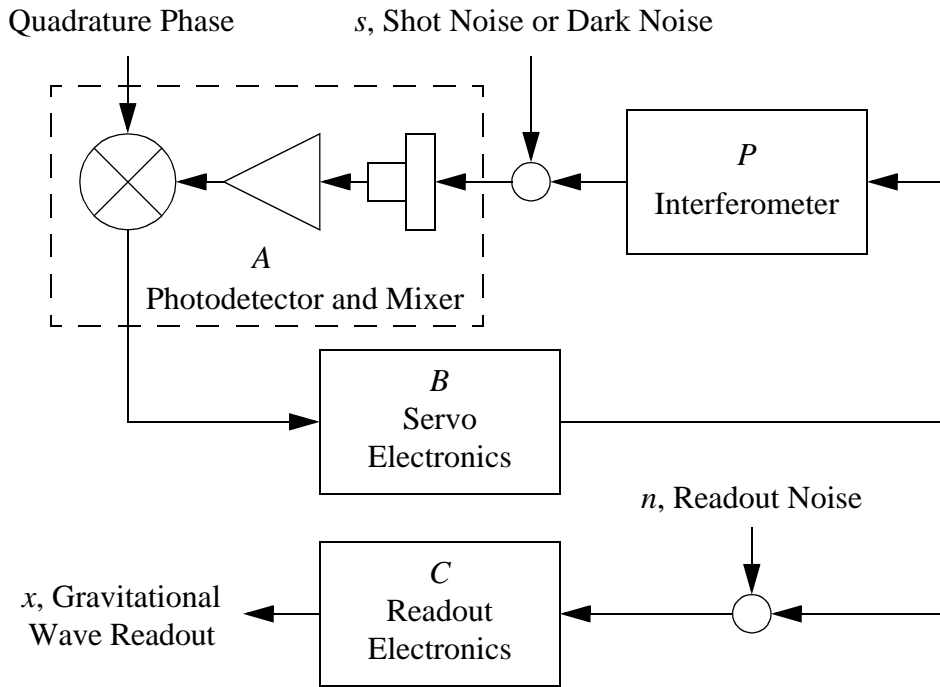


Figure 5