

Effect of parameter drift on Parametric Instability Threshold

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- Several studies to date (Phys Lett **A305**,111; **A354**,360) calculate the *threshold*, $R(\{\mathbf{p}\}) \geq 1$, of instability growth.
 - » All treat the contributing parameters $\{\mathbf{p}\}$ as fixed in time.
- May ask: will true dynamic system, $\{\mathbf{p}(t)\}$ alter threshold such that $R_{\text{eff}} < R(\langle\{\mathbf{p}(t)\}\rangle)$?
 - » Even slow drift may be suspect: $\tau_{PI} \sim \tau_m / (R-1) \sim 100\text{-}1000\text{s}$ in AdvLIGO for expected acoustic mode $\{m\}$ natural ring down $\sim 2\pi Q_m / \omega_m$
 - » Narrow acoustic resonances $\delta_m = \omega_m / Q_m$ allow very small drifts to slew many line widths in $\Delta t \ll \tau_m$
 - » AdvLIGO cavity mirror ROC change of $\sim 0.2\%$ (via thermal effects), HTM **PI** coupling modes can shift resonance ω_1 by several widths ($\sim \delta_1 / \pi = 250\text{ Hz}$)
- For Advanced Ligo $\{\mathbf{p}\}$, $R(\{\mathbf{p}_{\text{static}}\}) \approx 1$ for many acoustic $\{m\}$, so even small dilutions $R_{\text{eff}} < R(\{\mathbf{p}_{\text{static}}\})$ would be crucial.

Parameterization of **R** threshold

- Specific formulation: cavity field = sum_{i,j} over well defined transverse mode Lorentzian resonances

Each
Isolated Acoustic
Mode "m"

$$R = \frac{4P_0 Q_m}{mcL\omega_m^2} \left(\sum_i \frac{Q_{1i}\Lambda_{1i}}{1 + (\Delta\omega_{1i}/\delta_{1i})^2} - \sum_j \frac{Q_{1aj}\Lambda_{1aj}}{1 + (\Delta\omega_{1aj}/\delta_{1aj})^2} \right)$$

- Parameters:

P_0 = cavity circulating power

Q_m, ω_m = Acoustic mode Q , frequency

M = TM (mirror) mass

Q_{1j} = j^{th} cavity mode Q

Λ_{1j} = j^{th} cavity mode surface overlap with m

δ_{1j} = j^{th} cavity mode Lorentzian line width

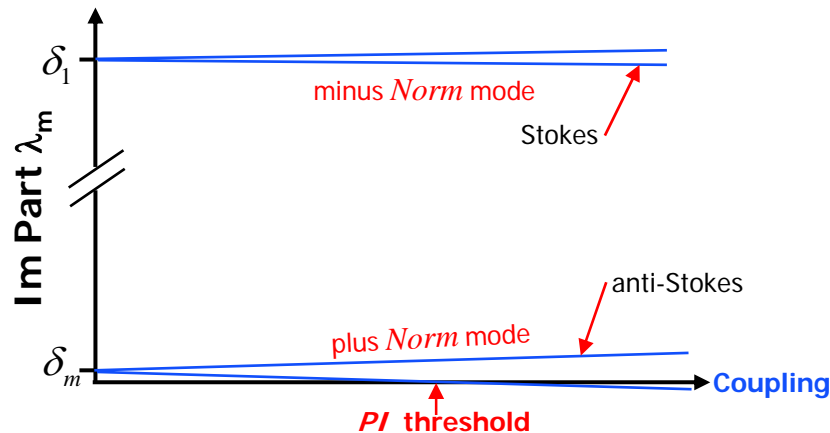
$\Delta\omega = \omega_0 - \omega_m - \omega_{\text{Gouy}}$ depends on detailed cavity geometry

$\Delta\omega_a = \omega_0 + \omega_m - \omega_{\text{Gouy}}$

- Only "Stokes" excitations at $\omega_{1j} = \omega_0 - \omega_m$ contribute to $R > 0$
 - Formula not apparently dependent on $\beta = \omega_{\text{pl}} - \omega_m$ (no acoustic Lorentzian factor) ??
 - All cavity fields are expressed in stationary limit: acoustic source and cavity parameters cannot change (t)

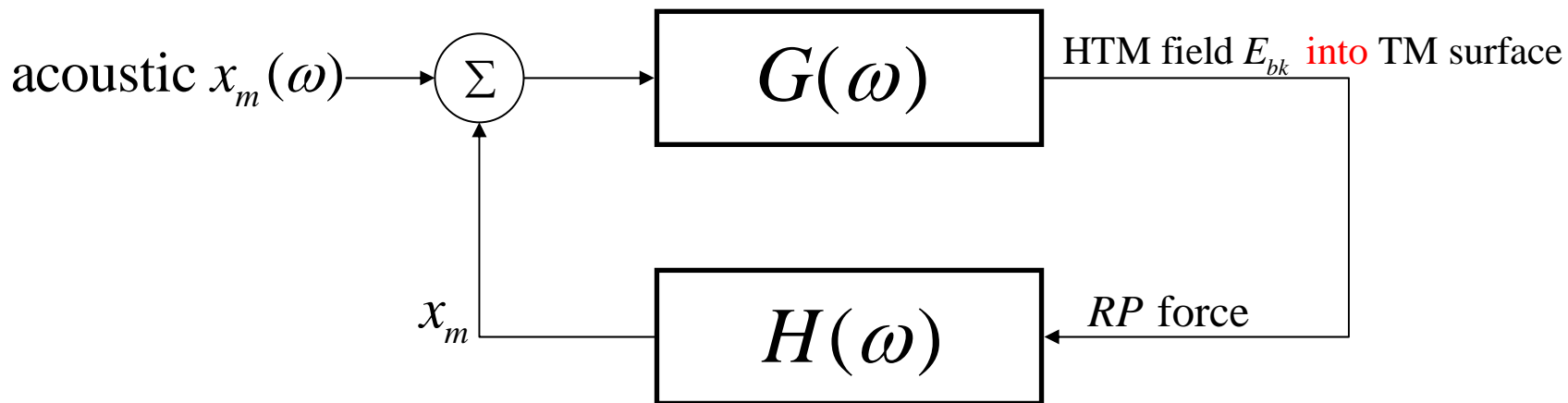
Nature of the *PI* “mode”

- Original concept (Phys. Lett A **287**, 331& **299**, 326) was eigenvalue, λ , solution to coupled, linearized 3 mode interaction equations.
 - » Normal modes of *coupled* acoustic + cavity SHOs
 - Coupling via 3^d mode, $E_0(\omega_0)$, but approximated as fixed parameter.
 - » Typical λ_m^\pm solutions correspond to two distinct normal modes (**NOT** Stokes, a-Stokes !)
 - » $\text{Im}[\lambda^-] < 0$ for any physical coupling: always damped.
 - However this normal mode is \sim free cavity HTM, so not of interest.
 - » $\text{Im}[\lambda^+] < 0$ in a-Stokes approximation. In Stokes $\text{Im}[\lambda^+] \rightarrow 0$ for sufficient coupling: unstable
 - » Thus only λ^+ mode of interest with $\text{Re}[\lambda^+] = \beta$ shift w.r.t. ω_m .



Feedback model

- **Linearized** coupled SHOs work well \Rightarrow equivalent Feedback analysis:



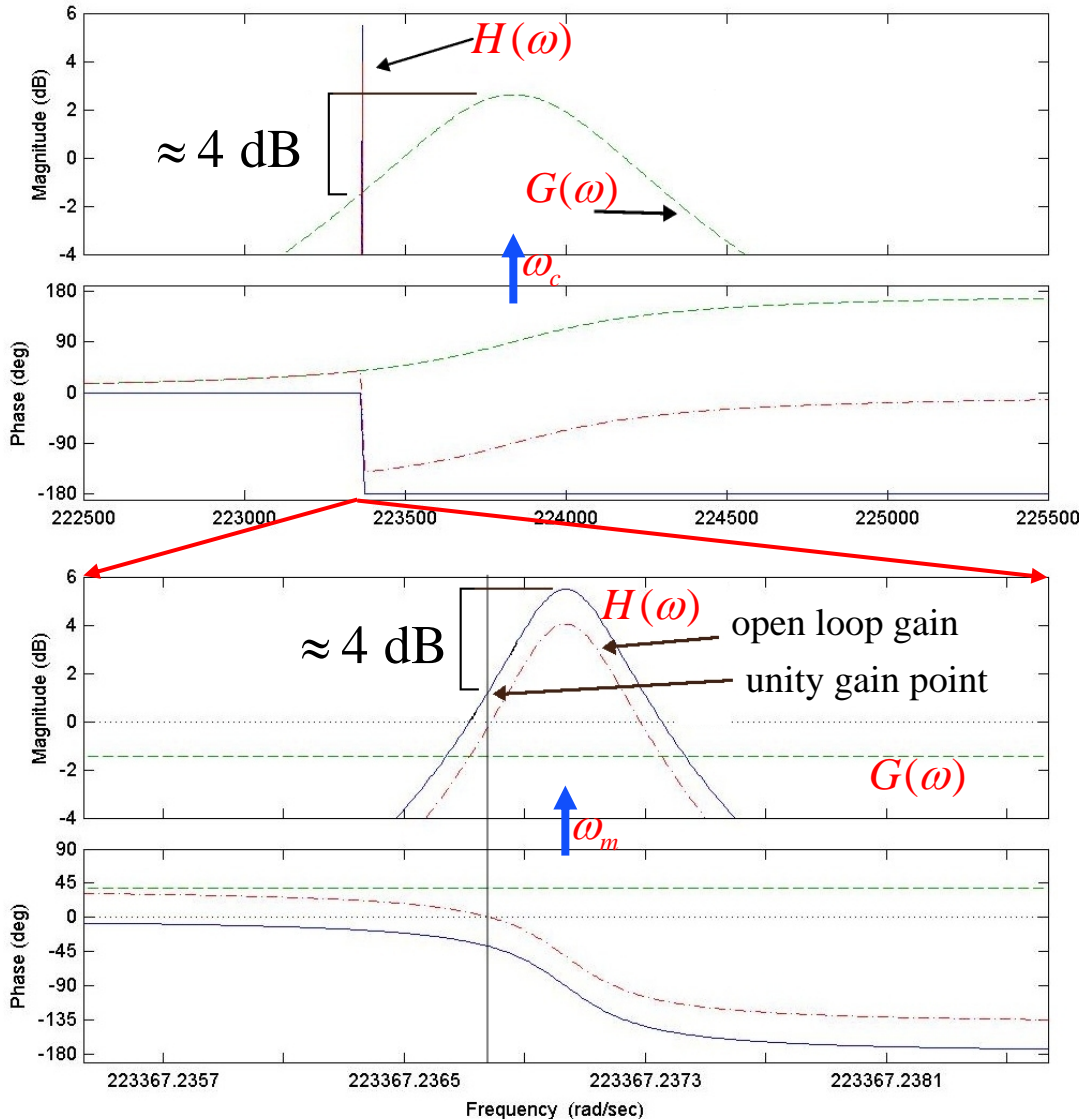
- Split into two transfer functions:
 - » TM acoustic Amplitude \rightarrow excited Stokes/a-Stokes HTM cavity field: $G(\omega)$
 - » Cavity field \rightarrow Force on TM Surface (radiation pressure): **Const.**
 - » Force \rightarrow driven response of damped acoustic oscillator: $H(\omega)$

$$G(\omega) = i\alpha \left(\frac{-e^{-i\left(\frac{-2\omega L}{c} + \phi_{as}\right)}}{1 - r_1 r_2 e^{-i\left(\frac{-2\omega L}{c} + \phi_{as}\right)}} + \frac{e^{i\left(\frac{2\omega L}{c} + \phi_s\right)}}{1 - r_1 r_2 e^{i\left(\frac{2\omega L}{c} + \phi_s\right)}} \right) \xrightarrow{\text{Single pole \& Stokes Approx.}} \frac{\alpha}{-\omega + \omega_c - \frac{1-r_1 r_2}{r_1 r_2} \frac{c}{2L} i}$$

$$H(\omega) = \frac{1}{-\omega^2 + i\gamma\omega + \omega_m^2} \xrightarrow{\text{Single pole \& Stokes Approx.}} \frac{1}{-\omega + \omega_m + i\frac{\gamma}{2}} \left(\frac{1}{2\omega_m} \right)$$

- $G(\omega)$ contains two terms: Stokes and anti-Stokes
 - » Stokes amplifies TM vibrations, anti-Stokes damps
 - » For well spaced cavity HTM resonances only one \sim coincides with $\omega_0 \pm \omega_m$
- Open loop *unity gain* ω_{UG} corresponds to $G(\omega_{UG}) H(\omega_{UG}) = R$
 - » Net phase shift = 0 means $G(\omega_{UG}), H(\omega_{UG})$ must be phase conjugates.
 - » Eigen-frequency coupled system occurs at equal magnitude decrements from the two transfer function peaks

Feedback Bode analysis



- Open loop transfer function has unity gain, with 0 phase shift. **Unstable!**
- At freq. where open loop gain = 1, equal factor down peaks in both transfer functions

Acoustic mode Energy gain/dissipation

- Rate of work, \dot{W} done on mirror by radiation pressure.
- Compared with natural dissipation rate, U_m/τ_m of acoustic {m}
 - » Since {m} strictly does no work on cavity field: pure *parametric* excitation.
- **Instability identified by $\dot{W} \tau_m / U_m \geq 1$** (Kells, LIGO-T060296)
 - » Independent of specific eigen-frequency, $\beta \ll \delta$. Therefore assume $\beta=0$.
- Not dependent on specific cavity field formulation ~~modal~~
 - » \dot{W} is real physical quantity: no phase condition, so β irrelevant
 - » Allows unambiguous meaning to $R(\{p\})$ away from threshold (LIGO-T060207)
 - » If field Approx. as modes: recover same coupled mode R formula, thus verifying previous interpretation as sum over modal parts

Drift in parametric spectrum

- Coupled mode PI analysis to date: necessarily static system $\{p\}$
- Anticipate significant drift in most $\{p\}$
 - » Drift of ω_m due to T_{ambient} , beam heat, stress, thermal deformation.
 - Especially as cavity power is cycled to $P_{0\text{max}}$
 - Changes to ω_{ij} due to cavity deformations (mirror radius, beam position)
 - Parametric induced change: β , τ_m

- Expected instability $\tau_{PI} \equiv \frac{\tau_m}{R-1} = \frac{2}{\delta_m (R-1)} \approx \frac{2}{.001 \text{s}^{-1} (< 10)} \approx 10^3 \text{ sec}$

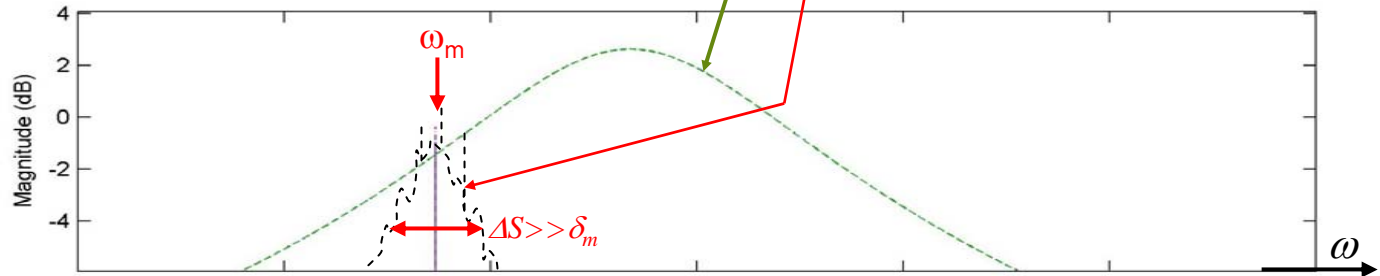
- Expected thermal ω_m drift $< 1 \text{ Hz/hr}$ so that $\tau_{PI} \cdot \text{drift rate} / \delta_m \gg 1$
 - » Static analysis assumes drift $< \delta_m$, β
 - » How do we know R_{static} reasonably represents drifting interferometer?

- Analysis via the linear feedback model resolves this:
 - » Fundamental problem is linear near threshold \Rightarrow invoke Fourier decomposition!
 - » Drift described in terms of broader ($line \gg \delta_m$) spectrum acoustic excitation
 - » Interpret power spectrum as $\propto \dot{W}$ [non-linear!] via Parseval theorem.

Work_m: spectrum shape independent

- Stokes cavity $E_{1j}(\omega_0 - \omega)$ components linear in acoustic $\{m\} x_m(\omega)$, using $x_m(\omega) \equiv \tilde{x}_m$

$$\tilde{E}_{bk} \equiv E_{bk}(\omega_0 - \omega) = G(\omega)\tilde{x}_m \approx G(\omega_m)\tilde{x}_m$$



- $\dot{W}(\omega)$ density acting on $\{m\}$ thus closely in proportion to spectral density of $x_m(\omega)$:

$$\dot{W}(\omega)d\omega \propto E_{bk}\omega\tilde{x}_m^*d\omega = G(\omega)\tilde{x}_m\omega\tilde{x}_m^*d\omega$$

- Then, by Parseval, mean rate of work done on $\{m\}$:

$$\int \dot{W}(\omega)d\omega \propto \int_{\Delta S} G(\omega)|\tilde{x}_m|^2\omega d\omega \approx G(\omega_m)\omega_m \int_{\Delta S} |\tilde{x}_m|^2 d\omega \propto \langle U_m(t) \rangle_{\Delta t \sim \Delta S^{-1}}$$

that is, independent of particular spectrum of $x_m(\omega)$

No PI dilution

- *Physical* assumption: that $U_m(t)$ cannot change due to parameter drift
 - » E.g. isolated TM energy is conserved (excepting slow dissipation).
- Condition for this independence: $\delta_m \sim .002\text{rad/s} \leq \Delta S \ll \delta_{\text{Cav mode}} > 500\text{rad/s}$
 - » For example: LIGO TM “ambient” drift in ω_m is $< 1\text{Hz/hr}$
 - » Faster dithering ω_m ? Strictly limited by smooth oscillator $U_P \leftrightarrow U_K$ continuity any short time SHO phase jumps \ll up conversion scale.
- Conclusion: plausible TM drifts, even by $\gg \delta_m$, can't dilute PI
- Another category: cavity parameter (Guoy phase) change effect on $H(\omega)$.
 - » $H(\omega)$ treated as a static filter..... to be modified in dynamic reality.
 - » Static Approx. holds if $\tau_{1j} \Delta\omega_{1j} < \pi$ which holds for $\Delta\omega_{1j} < \delta_{1j}$ ($\sim 2\pi \cdot 85$ Hz) during $t \sim \tau_{1j} < 10\text{ms}$.