Proposal
Thermal and Thermoelastic Noise Research for Advanced LIGO Optics

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Outline

- Objective and Approach
- Output of the Prior NSF Support
  - Laser phase noise formulas for optical resonator and delay line
  - Coating noise estimation
- Proposed Work
  - Coating noise studies for mirrors of edge geometry
  - Thermo-elastic noise of coated mirrors
  - Thermal & thermo-elastic noises of realistic mirror designs
- Tasks & Time Lines
- Summary
- (Broader Impacts)

Objective and Approach

Objectives

- To develop laser-phase-noise formulas
  - Green's-function-based
  - Analytical and computational
  - two-point laser-phase-fluctuation correlations
  - complex test mass objects.
- To extend the noise estimation method to thermo-elastic noises.
- To estimate thermal and thermo-elastic noises of coated mirrors for advanced LIGO designs.
- To examine merits of interferometer design options for future LIGO.

Approaches

- Calculate phase noise via Green's function method
  - Elasticity
  - Thermo-elasticity & thermal diffusion
- Analytical mirror models
  - Half space
  - Quarter space
  - Thin coating layer
- Numerical calculations
  - Realistic mirror shapes
  - Coating loss
  - Delay-line vs. Fabry Perot

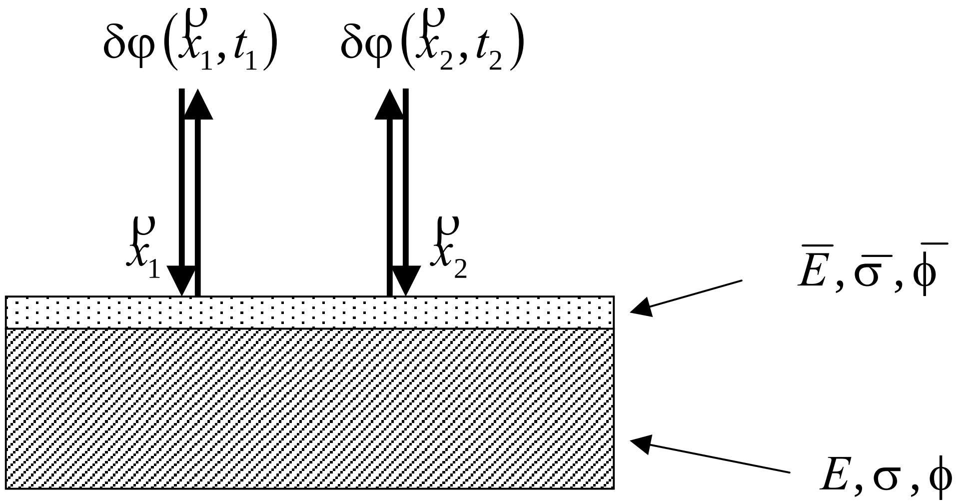
Output to Date

- Publications
  - N. Nakagawa, Eric Gustafson, P. Beyersdorf and M. M. Fejer "Estimating the off resonance thermal noise in mirrors, Fabry-Perot interferometers and delay-lines: the half infinite mirror with uniform loss," to appear in Phys. Rev. D
  - N. Nakagawa, A. M. Gretarsson, E.K. Gustafson, and M. M. Fejer, "Thermal noise in half infinite mirrors with non-uniform loss: a slab of excess loss in a half infinite mirror," submitted to Phys. Rev. D
  - N. Nakagawa, E.K. Gustafson, and M. M. Fejer, "Thermal phase noise estimations for fabry-perot and delay-line interferometers using coated mirrors," in preparation.
  - D. Crooks, et al., "Excess mechanical loss associated with dielectric mirror coatings on test masses in interferometric gravitational wave detectors," submitted to Classical and Quantum Gravity.
  - Gregory M. Harry, et al., "Thermal noise in interferometric gravitational wave detectors due to dielectric optical coatings," submitted to Classical and Quantum Gravity.

Phase-Noise Correlation

- Requirement
  - Compute laser phase noise correlation

$$\left\langle \delta\varphi\left(\vec{x}_1, t_1\right)\delta\varphi\left(\vec{x}_2, t_2\right)\right\rangle = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega\left(t_1 - t_2\right)} S_\varphi\left(\omega, \vec{x}_1, \vec{x}_2\right)$$



Ex. Coating noise model

Intrinsic Thermal Noise

- Phase Noise Formula

$$S_\varphi\left(\omega,\vec{r}_1,\vec{r}_2\right)=4k^2\,\frac{2k_BT}{\omega}\iint dS'\int dS''\,\psi_{00}^{w}\left(\vec{r}\,'-\vec{r}_1\right)\psi_{00}^{w}\left(\vec{r}\,''-\vec{r}_2\right)$$
$$\times\int_V d^3x\left[\partial_i\chi_{nj}^{\omega}\left(\vec{x},\vec{r}\,';c'\right)\right]c''_{ijkl}\left(\vec{x}\right)\left[\partial_k\chi_{nl}^{\omega}\left(\vec{x},\vec{r}\,'';c'\right)\right]^*$$

$$c_{ijkl}^{\omega}=c'_{ijkl}-ic''_{ijkl}\approx\left[1-i\phi\left(\omega\right)\right]c'_{ijkl}$$

$S_\varphi\left(\omega,\vec{r}_1,\vec{r}_2\right)$	the two-point laser-beam phase-noise power-spectrum correlation
$\left\langle u_i\left(\vec{r}\,'\right)u_j\left(\vec{r}\,''\right)\right\rangle_\omega$	the displacement spectral correlation
$\vec{r}_1,\vec{r}_2$	The laser beam reflection points (the beam centers)
$\psi_{00}^{w}\left(\vec{r}\right)\left(\propto e^{-2\lvert\vec{r}\rvert^2/w^2}\right)$	the Gaussian laser-beam profile function

w, k	the laser beam spot size (amplitude radius), and wave number
χ_{ij}^{ω}	elastic Green's function
c_{ijkl} [c'_{ijkl}, c''_{ijkl}]	elastic constants [dispersive and absorptive parts]
$\phi(\omega)$	loss function
k_B , T	Boltzmann constant, and temperature

Fluctuation-Dissipation Relation

- Surface Force density ➔ Strain

$$F\psi_{00}^{w}\left(\vec{r}' - \vec{r}_1\right) \Rightarrow u_{ij}^{\omega}\left(\vec{x};\vec{r}_1\right) = F\iint dS'\left[\partial_i \chi_{nj}^{\omega}\left(\vec{x},\vec{r}';c'\right)\right]\psi_{00}^{w}\left(\vec{r}' - \vec{r}_1\right)$$

$$S_{\varphi}\left(\omega,\vec{r}_1,\vec{r}_2\right) = 4k^2\,\frac{2k_BT}{\omega}\,\frac{1}{F^2}\int_V d^3x\left[u_{ij}^{\omega}\left(\vec{x};\vec{r}_1\right)\right]c''_{ijkl}\left(\vec{x}\right)\left[u_{kl}^{\omega}\left(\vec{x};\vec{r}_2\right)\right]^*$$

- Laudau-Lifshitz

$$\overline{\dot{E}_{mech}} = \underbrace{-\left(\kappa/T_0\right)\int_V dV\,\overline{\left(\vec{\nabla}T\right)^2}}_{thermo-elastic}\;\underbrace{-\,\frac{\omega^2}{2}\int_V dV\,u_{ij}^{\omega}\,\eta_{ijkl}\,u_{kl}^{\omega\,*}}_{viscous}, \quad c''_{ijkl} = \underbrace{\omega\eta_{ijkl}}_{viscous} + \underbrace{\Lambda}_{structural}$$

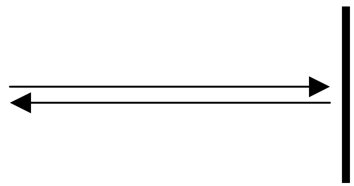
$$\boxed{S_{\varphi}\left(\omega,\vec{r},\vec{r}\right) = 4k^2\,\frac{4k_BT}{\omega^2}\,\frac{1}{F^2}\left(-\overline{\dot{E}_{mech}}\right)}$$

Various Optical Configurations

- Phase noise formulas

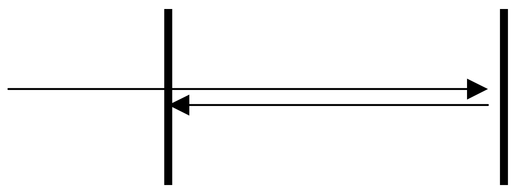
  Computed explicitly for

  – Single-reflection mirror



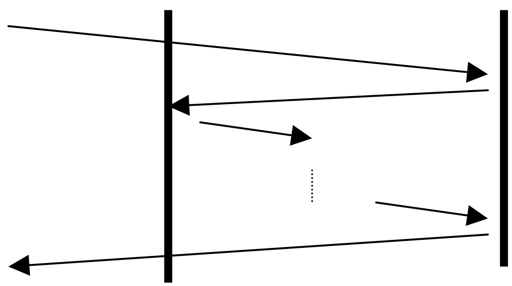
$$S_\varphi^{Single}(\omega) = S_\varphi(\omega, \vec{r}, \vec{r})$$

  – Fabry-Perot resonator



$$S_\varphi^{FP}(\omega) = \left[\frac{(1+r_I)^2}{1+r_I^2}\right]\left[1 - \frac{2r_I}{1+r_I^2}\cos 2\omega\tau\right]^{-1}\left[S_\varphi^E(\omega) + r_I^2 S_\varphi^I(\omega)\right]$$

  – Optical delay line



$$S_\varphi^{DL}(\omega) = \sum_{n=1}^{N} S_\varphi^E(\omega, \vec{r}_n, \vec{r}_n) + 2\sum_{n=2}^{N}\sum_{q=1}^{n-1}\cos[2(n-q)\tau\omega]\cdot S_\varphi^E(\omega, \vec{r}_n, \vec{r}_q)$$
$$+\sum_{n=1}^{N-1} S_\varphi^I(\omega, \vec{\rho}_n, \vec{\rho}_n) + 2\sum_{n=2}^{N-1}\sum_{q=1}^{n-1}\cos[2(n-q)\tau\omega]\cdot S_\varphi^I(\omega, \vec{\rho}_n, \vec{\rho}_q)$$

r_I	the input mirror reflection coefficient
τ	the transit time
$S_\varphi^E(\omega)$, $S_\varphi^I(\omega)$	the single-reflection phase noises of the input and end-point mirrors.
$\vec{r}_n$	the positions of the N-time reflections on the end-mirror surface
$\vec{\rho}_p$	the positions of the (N-1)-time reflections on the input-mirror surface
E, σ	Young's modulus, and Poisson ratio

If Half space:

$$S_\varphi(\omega, \vec{r}_1, \vec{r}_2) = \frac{8k_B T}{\sqrt{\pi}}\frac{\phi}{\omega}\frac{k^2}{w}\frac{1-\sigma^2}{E} e^{-(\vec{r}_1-\vec{r}_2)^2/2w^2} I_0\left((\vec{r}_1-\vec{r}_2)^2/2w^2\right)$$

Fabry-Perot vs. Delay Line

- Fabry-Perot vs Delay lines
  - Analytical half-space mirror model
  - Fabry-Perot interferometer vs several delay lines
  - Storage time as proposed for LIGO II.
  - Delay line beam centers
    - evenly spaced
    - on a circle
- When the spots are not overlapping appreciably, the delay line is less noisy than the Fabry-Perot.
  - Noise levels are similar if
    - the spot circle radii comparable to the beam spot size
    - the spots are largely overlapping, and above several hundred Hertz.

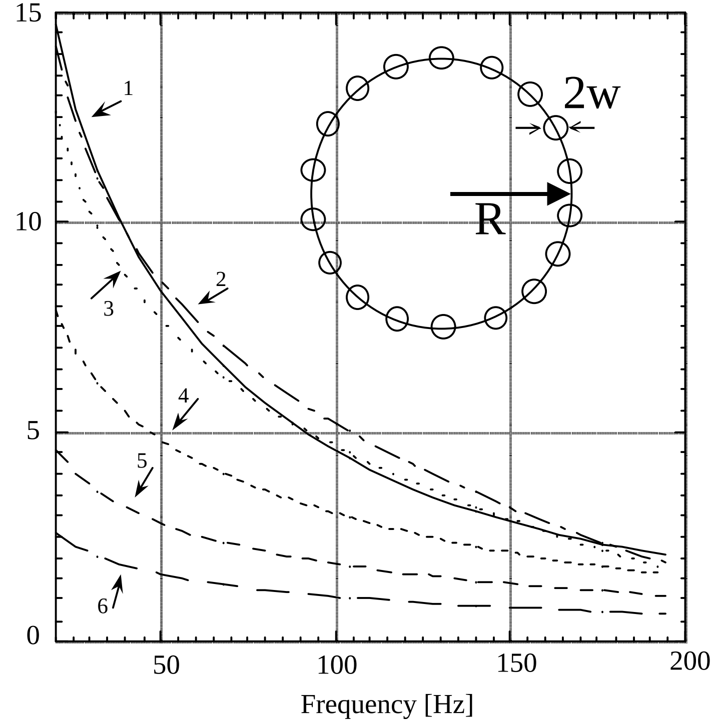
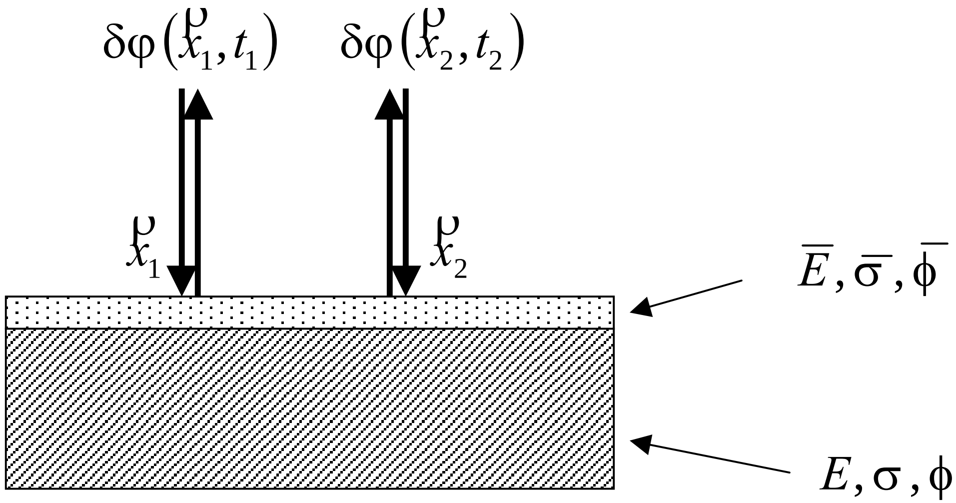


Figure 1: Comparison of the phase noise from a delay-line and a Fabry-Perot interferometer. The solid curve (1) is for a 4 km Fabry-Perot interferometer with an input mirror power reflectivity of RI=0.97 and an end mirror power reflectivity of RE=1.00. Curves 2, 3, 4, 5 and 6 correspond to 4 km delay lines all with 130 spots on the end mirror and laser beam spots of 1/e field radius w in a pattern with their centers on a circles of radius R=w/3 (2), R=2w/3 (3), R=5w/2 (4), R=10w (5) and R=20w (6) where w=3.5 cm, the spot size used for both mirrors of the Fabry-Perot interferometer. The mirror Q is assumed to be 3´108 and the material properties are those of Sapphire E=71.8 GPa and s=0.16 however we are treating sapphire as isotropic for the purpose of this illustration and assuming a single loss function.

Coating Noise; Problem Statement

- Coating noise model
  - Based on half-space mirror model
    - a lossy layer (thickness d) on a lossy host material
  - Requirement: Compute laser phase noise correlation
  - Approach: via analytical Green's function



$$\left\langle \delta\varphi(\vec{x}_1, t_1)\,\delta\varphi(\vec{x}_2, t_2) \right\rangle$$
$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t_1 - t_2)} \left\langle \delta\varphi(\vec{x}_1)\,\delta\varphi(\vec{x}_2) \right\rangle_\omega$$
$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t_1 - t_2)} S_\varphi(\omega, \vec{x}_1 - \vec{x}_2)$$

Coating Noise

- Static Green's function for layer-on-substrate

$$\left[\chi^{sos}_{\cdot\cdot}\left(\tilde{p},z\right)\right]=\frac{1}{p\bar{E}}\,\frac{1+\bar{\sigma}}{1-\bar{\sigma}}\left\{\begin{aligned}&\left[\left(2\left(1-\bar{\sigma}\right)-pz\cdot i\tau_1\right)G^{sos}\left(\tilde{p},0\right)+\tfrac{1}{2}\,pz\tau_3\right]\cosh pz\\&+\left[\left(pz\tau_3+\left(1-2\bar{\sigma}\right)\tau_2\right)G^{sos}\left(\tilde{p},0\right)+\tfrac{3-4\bar{\sigma}}{2}-\tfrac{1}{2}\,pz\left(i\tau_1\right)\right]\sinh pz\end{aligned}\right\}$$

$$\begin{aligned}\left[\Phi^{sos}_{z\cdot\cdot}\left(\tilde{p},z\right)\right]=&\;\tfrac{1}{2\left(1-\bar{\sigma}\right)}\left[2\,pz\tau_3G^{sos}\left(\tilde{p},0\right)+2\left(1-\bar{\sigma}\right)-pz\cdot i\tau_1\right]\cosh pz\\&+\tfrac{1}{2\left(1-\bar{\sigma}\right)}\left[2\left[1-pz\left(i\tau_1\right)\right]G^{sos}\left(\tilde{p},0\right)+pz\tau_3-\left(1-2\bar{\sigma}\right)\tau_2\right]\sinh pz\end{aligned}$$

where

$$G^{sos}\left(\tilde{p},0\right)=\Delta^{-1}\left\{\begin{aligned}&\alpha\left(1-\sigma\right)\cosh 2pd+\tfrac{1}{4\left(1-\bar{\sigma}\right)}\left\{\tfrac{3-4\bar{\sigma}}{2}+\alpha\left(1-2\sigma\right)\left(1-2\bar{\sigma}\right)+\tfrac{\alpha^2}{2}\left(3-4\sigma\right)\right\}\sinh 2pd\\&+\left\{\begin{aligned}&\tfrac{\alpha}{2}\left[\left(1-2\sigma\right)\cosh 2pd+\tfrac{1-\sigma}{1-\bar{\sigma}}\left(1-2\bar{\sigma}\right)\sinh 2pd\right]\\&+\tfrac{1}{8\left(1-\bar{\sigma}\right)^2}\left[\left(1-2\bar{\sigma}\right)\left(3-4\bar{\sigma}\right)-2\alpha\left(1-2\sigma\right)\left(3-4\bar{\sigma}\right)+\alpha^2\left(1-2\bar{\sigma}\right)\left(3-4\sigma\right)\right]\sinh^2 pd\\&-\tfrac{1}{8\left(1-\bar{\sigma}\right)^2}\left(1-\alpha\right)\left[1+\alpha\left(3-4\sigma\right)\right]\left(pd\right)^2\end{aligned}\right\}\tau_2\\&+\tfrac{1}{4\left(1-\bar{\sigma}\right)}\,pd\left(1-\alpha\right)\left[1+\alpha\left(3-4\sigma\right)\right]\tau_3\end{aligned}\right\}$$

$$\begin{aligned}\Delta\equiv&\left\{\cosh pd-\tfrac{1}{2\left(1-\bar{\sigma}\right)}\left[\left(1-2\bar{\sigma}\right)-\alpha\left(3-4\sigma\right)\right]\sinh pd\right\}\left\{\cosh pd+\tfrac{1}{2\left(1-\bar{\sigma}\right)}\left[\left(1-2\bar{\sigma}\right)+\alpha\right]\sinh pd\right\}\\&+\tfrac{1}{4\left(1-\bar{\sigma}\right)^2}\left(1-\alpha\right)\left\{1+\alpha\left(3-4\sigma\right)\right\}\left(pd\right)^2\end{aligned}$$

$\alpha\equiv\frac{1+\sigma}{1+\bar{\sigma}}\frac{\bar{E}}{E},\quad \tau_{1,2,3}=\text{Pauli matrices}$

D. M. Burmister, J. Appl. Phys. 16, 89-94, 1945.

Coating Noise

- Intrinsic Thermal Phase-Noise Estimation

$$S_\varphi^{coating}(\omega,\vec{r}) = 4k^2 \cdot \frac{2k_B T}{\omega} \frac{1}{\sqrt{\pi}\, w}$$
$$\times \left\{ \phi \cdot \frac{1-\sigma^2}{E} e^{-r^2/2w^2} I_0\left(r^2/2w^2\right) + \left[\bar{\phi} \frac{(1-2\bar{\sigma})(1+\bar{\sigma})}{1-\bar{\sigma}} \frac{1}{\bar{E}} + (\bar{\phi} - 2\phi) \frac{(1+\sigma)^2(1-2\sigma)^2}{1-\bar{\sigma}^2} \frac{\bar{E}}{E^2} - \phi \frac{2(1+\sigma)(1-2\sigma)\bar{\sigma}}{1-\bar{\sigma}} \frac{1}{E} \right] \frac{d}{\sqrt{\pi}\, w} e^{-r^2/w^2} + O\left(d^2/w^2\right) \right\}$$

$S_\varphi^{coating}(\omega,\vec{r})$	The phase noise two-point correlation for a coated half-space mirror; double-sided
E, σ, ϕ	Young's modulus, Poisson ratio, and loss function of the substrate material
$\bar{E}, \bar{\sigma}, \bar{\phi}$	Those of the coating material
d	The coating thickness
$I_0(z)$	The 0-th order modified Bessel function of the first kind; $I_0(0)=1$

$\omega = 2\pi f$	Frequency
$\vec{r}$	a relative position vector between the two beam centers on the coating surface; $\vec{r} = 0$ for a single reflection.
k, w	The laser beam wave number, and spot size (amplitude radius)
k_B, T	The Boltzmann constant and the temperature

Coating Noise

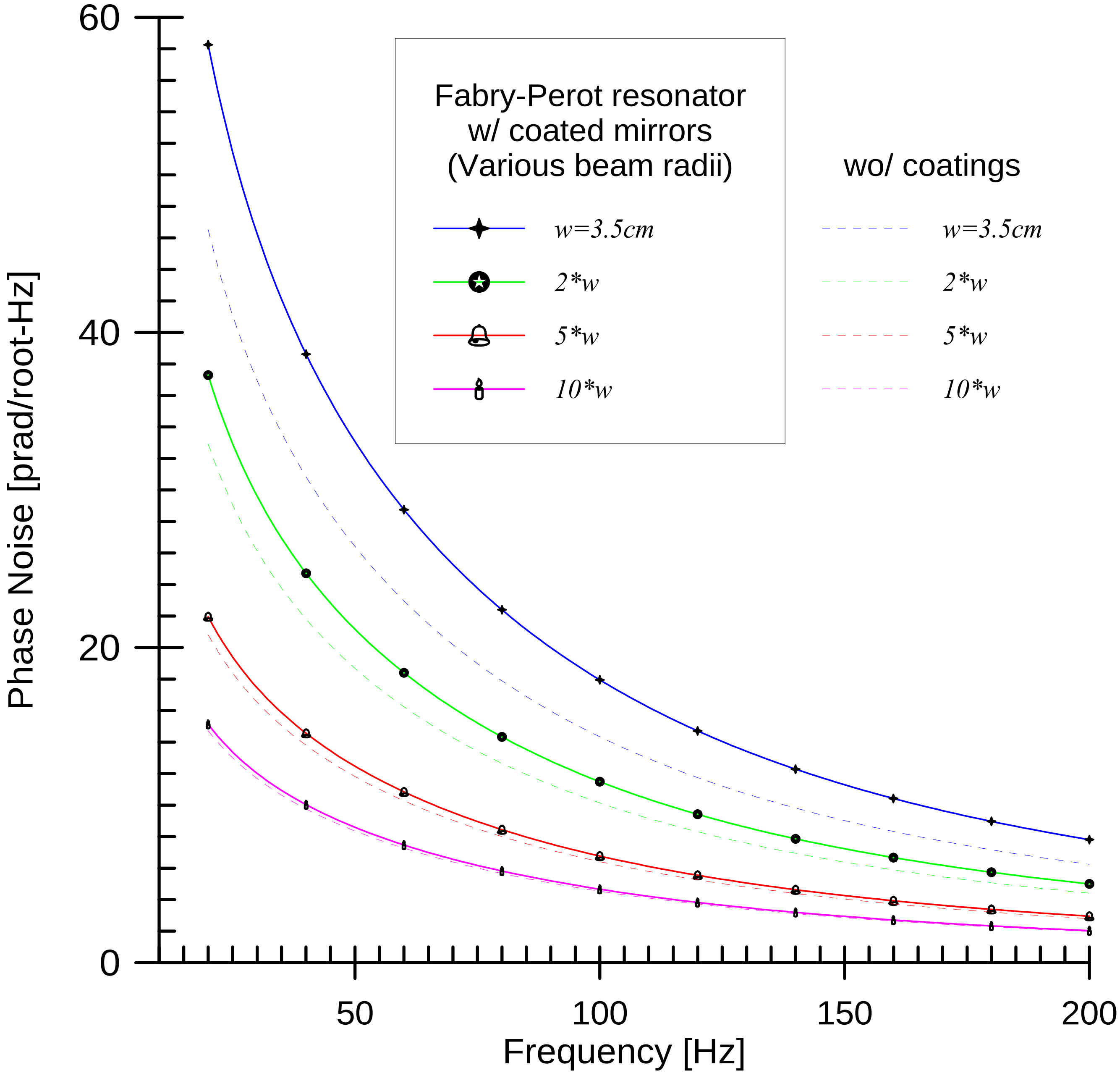
- Resonator

$$S_{\varphi}^{coating}(\omega,0)=4k^{2}\cdot\frac{2k_{B}T}{\omega}\underline{\frac{1}{\sqrt{\pi}\,w}}\left\{\phi\cdot\frac{1-\sigma^{2}}{E}+\left[\bar{\phi}\,\frac{(1-2\bar{\sigma})(1+\bar{\sigma})}{1-\bar{\sigma}}\,\frac{1}{\bar{\bar{E}}}+\Lambda\right]\underline{\underline{\frac{d}{\sqrt{\pi}\,w}}}\right\}$$

- Delay lines

$$S_{\varphi}^{coating}(\omega,\vec{r})=4k^{2}\cdot\frac{2k_{B}T}{\omega}\frac{1}{\sqrt{\pi}\,w}$$

$$\times\left\{\phi\cdot\frac{1-\sigma^{2}}{E}\underbrace{e^{-r^{2}/2w^{2}}I_{0}\left(r^{2}/2w^{2}\right)}_{\xrightarrow{r\to\infty}\frac{1}{r}}+\left[\bar{\phi}\,\frac{(1-2\bar{\sigma})(1+\bar{\sigma})}{1-\bar{\sigma}}\,\frac{1}{\bar{\bar{E}}}+\Lambda\right]\frac{d}{\sqrt{\pi}\,w}\underline{\underline{e^{-r^{2}/w^{2}}}}\right\}$$

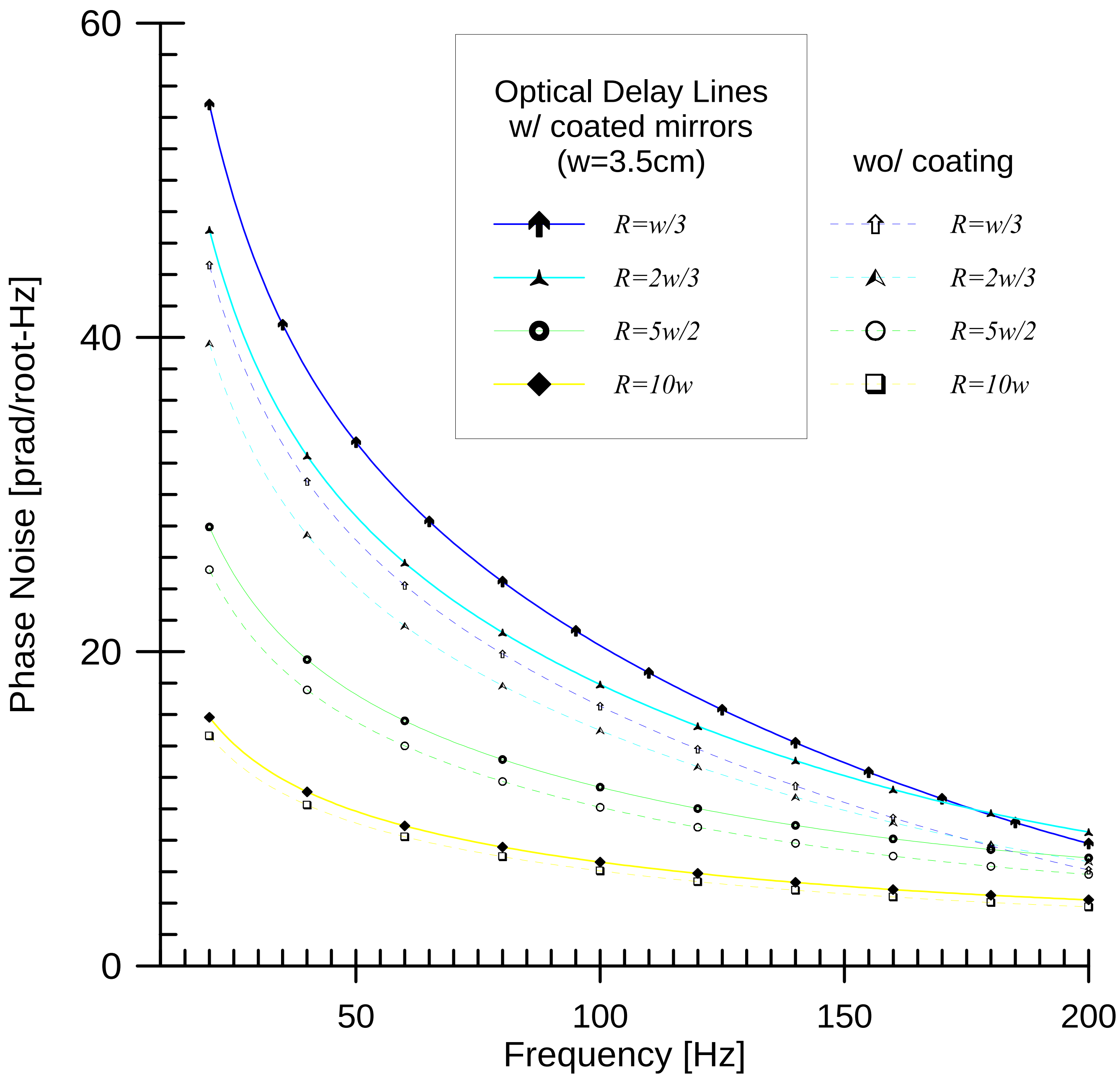


Substrate=Fused silica

$$\begin{cases} E = 72.6\ GPa \\ \sigma = 0.16 \\ Q = 3\times 10^{7} \end{cases}$$

Coating= average of Al_2O_3 and Ta_2O_5 (Crooks et al.)

$$\begin{cases} E = 260\ GPa \\ \sigma = 0.26 \\ Q = 1.6\times 10^{4} \end{cases}$$

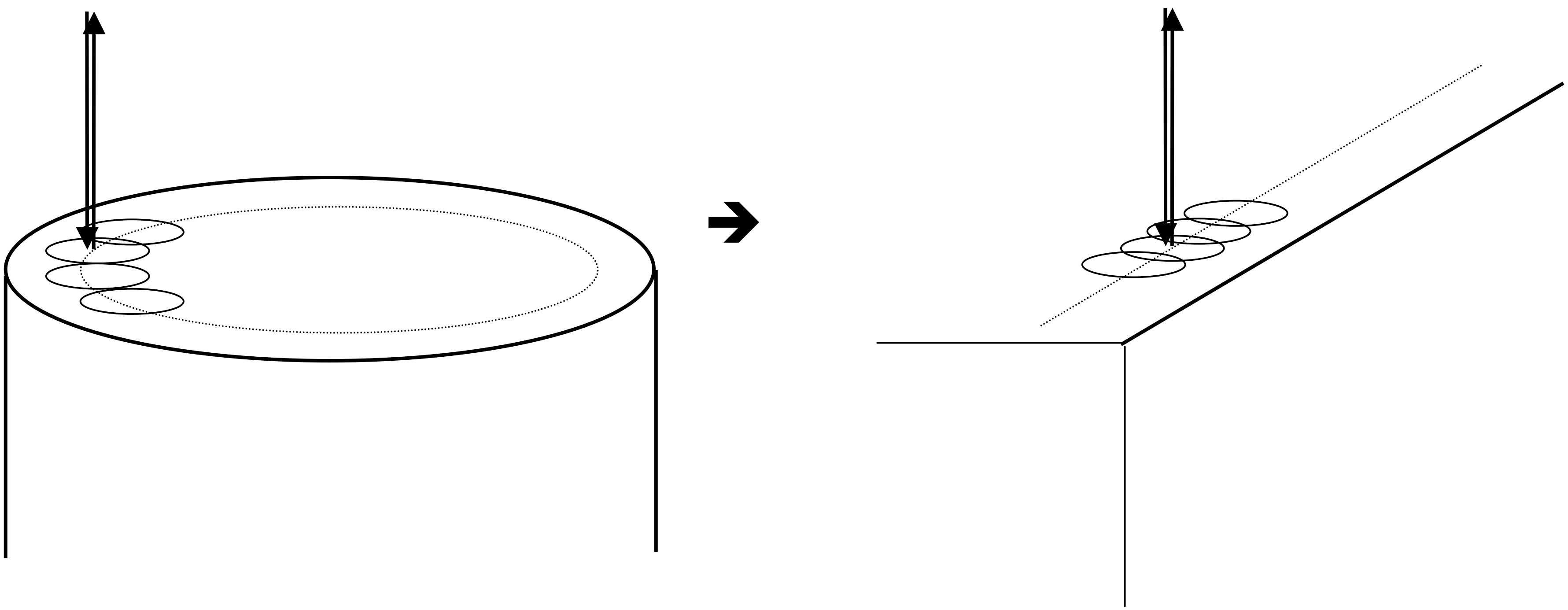

Optical Delay Lines
w/ coated mirrors
(w=3.5cm)
wo/ coating
R=w/3
R=2w/3
R=5w/2
R=10w
R=w/3
R=2w/3
R=5w/2
R=10w
Phase Noise [prad/root-Hz]
60
40
20
0
50
100
150
200
Frequency [Hz]

Proposed Work

- Edge effects on coated mirror noise
  - Analytical quarter-space model
- Extension to thermo-elastic noise
  - Coated mirrors
  - Delay-line interferometers
  - Mirrors at low temperatures
- Thermal & thermo-elastic noises
  - Mirrors of realistic shapes
    - Numerical Green's function calculation

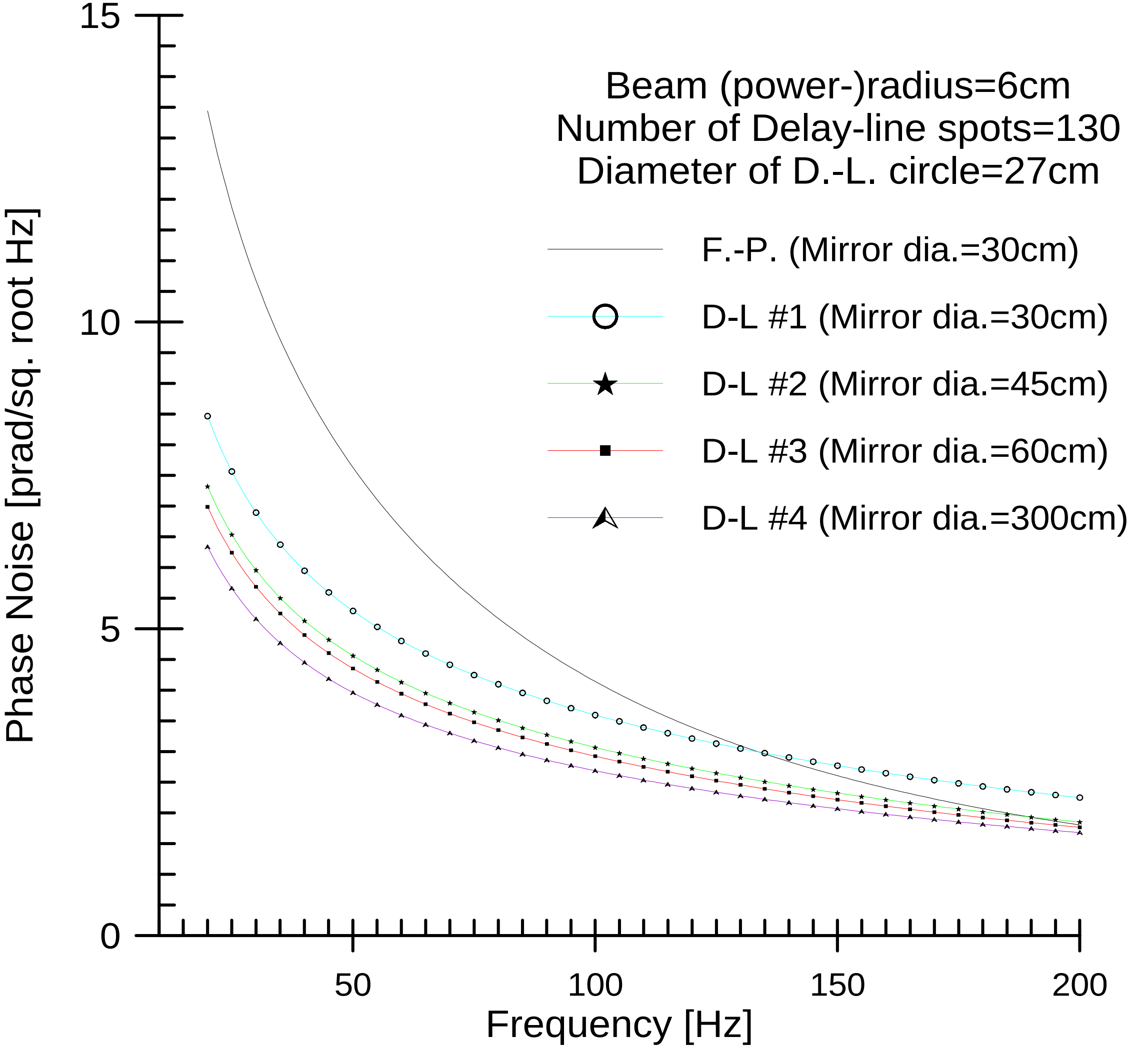
Quarter-Space Mirror Model

- Edge effects on noise calculation
  - Analytical model



Quarter-Space Mirror Model (con't)

- Finite-mirror size effect on thermal noise (scalar model)



Thermo-elasticity & Thermal diffusion

- From previous identification

$$S_\varphi\left(\omega,\vec r,\vec r\right)=4k^2\frac{4k_BT}{\omega^2}\frac{1}{F^2}\left(-\overline{\dot E_{mech}}\right)\qquad \dot E_{mech}=-\left(\kappa/T_0\right)\int_V dV\left(\vec\nabla T\right)^2-\int_V dV\,\dot u_{ij}\,\eta_{ijkl}\,\dot u_{kl}$$

$$\overline{\dot E_{mech}}=-\left(\kappa/T_0\right)\int_V dV\,\overline{\left(\vec\nabla T\right)^2}-\frac{\omega}{2}\int_V dV\,u^\omega_{ij}\,c''_{ijkl}\,u^{\omega\,*}_{kl}$$

- If the adiabatic condition holds

$$T-T_0=-\frac{T_0\rho\alpha}{C_p}\left(v_l^2-v_t^2\right)u_{ll}$$

$$S_\varphi^{th-el}\left(\omega,\vec r_1,\vec r_2\right)=4k^2\frac{2k_BT}{\omega}\iint dS'\int dS''\,\psi^{w}_{00}\left(\vec r'-\vec r_1\right)\psi^{w}_{00}\left(\vec r''-\vec r_2\right)$$
$$\times\frac{\kappa}{\omega}T\left[\frac{\rho\alpha}{C_v}\left(v_l^2-v_t^2\right)\right]^2\int_V d^3x\left[\partial_i\partial_l\chi^\omega_{nl}\left(\vec x,\vec r';c'\right)\right]\left[\partial_i\partial_k\chi^\omega_{nk}\left(\vec x,\vec r'';c'\right)\right]^*$$

Numerical elasticity

- Noise study:
  Cylindrical mirrors
  - Numerical Green's function computation
    - Betti-Rayleigh-Somigliana formula
    - Nodal discretization by the boundary element method.
  - Cylindrical mirror model
    - Single reflection
    - Fabry-Perot
    - Delay-line
  - Demonstrate the delay-line vs. resonator assertion
  - Effects of mirror aspect ratios.

$$-\rho\omega^2 u_i - \partial_j T_{ij} = 0$$

$$-\rho\omega^2 \Gamma_{ik} - \partial_j \Phi_{ijk} = \delta_{ik}\delta\left(\vec{x} - \vec{x}_P\right)$$

$$\Rightarrow \chi^V\left(\vec{x}_P\right) u_k\left(\vec{x}_P\right) = \int_S dS_j \left[-u_i \Phi^{\omega}_{ijk} + T_{ij}\Gamma^{\omega}_{ik}\right]$$

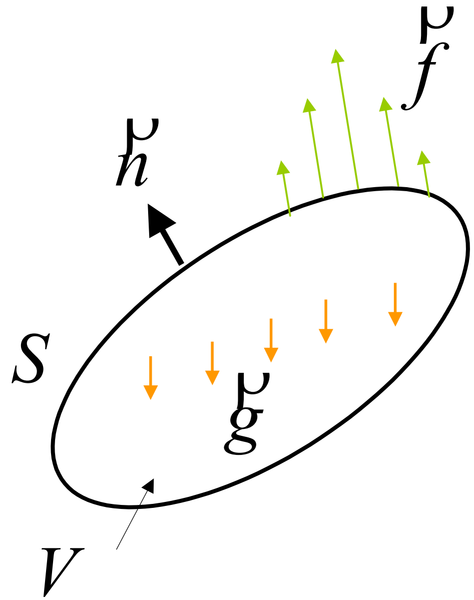
$u_i(\vec{x})$	Displacement
$T_{ij}(\vec{x})$	stress tensor $\left(T_{ij} \equiv c_{ijkl}\partial_k u_l\right)$
$\Gamma^{\omega}_{ij}(\vec{x})$	the fundamental solution (Green's function)
$\Phi^{\omega}_{ijk}(\vec{x})$	$\equiv c_{ijlm}\partial_l \Gamma_{mk}$
c_{ijkl}	elastic constants
ρ	Density
V, S	volume of a region and its boundary surface
$\chi^V(\vec{x})$	the characteristic function of V (=1 inside V, =0 outside)

Static Elasticity

- Needs to avoid rigid-body motions
  - Mass and moment of inertia

$$M = \int_V dV\,\rho, \quad I_{ij} = \int_V dV\,\rho\left(x^2\delta_{ij} - x_i x_j\right)$$

  - Given surface forces *f* can be counter-balanced by volume forces



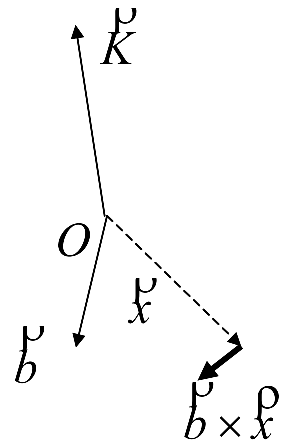
$$\vec{g}(\vec{x}) = \vec{a} + \vec{b}\times\vec{x}, \quad \begin{cases} \vec{a} \equiv -\frac{1}{V}\vec{F} \\ \vec{b} \equiv -\frac{M}{V}\overline{\overline{I}}^{-1}\cdot\vec{K} \end{cases}$$

where

$$\vec{F} \equiv \int_S dS\,\vec{f}(\vec{x}), \quad \vec{K} \equiv \int_S dS\,\vec{x}\times\vec{f}(\vec{x})$$

$$\vec{F}_g \equiv \int_V dV\,\vec{g}(\vec{x}), \quad \vec{K}_g \equiv \int_V dV\,\vec{x}\times\vec{g}(\vec{x})$$

$$\vec{F} + \vec{F}_g = 0, \quad \vec{K} + \vec{K}_g = 0$$



Tasks & Time Lines

T1. Coating noise estimation
- spatial de-correlation

T2. Numerical noise estimations
- Realistic mirror shapes
- Validation

T3. Quarter space
- Analytical model
- Finite size effect

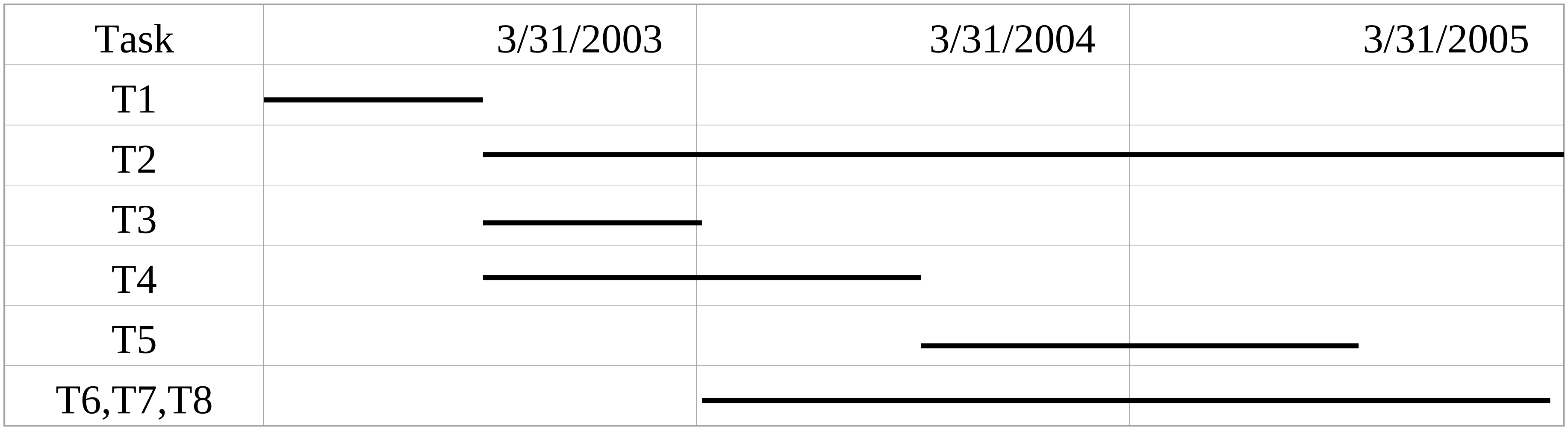
T4. Thermo-elastic noise estimation
- delay-line vs. resonator
- coating

T5. Quarter space II
- coating

T6. Anisotropy effects

T7. Dielectric loss study

T8. Off-beam-axis scatterings



Summary

Accomplishments

- Obtained two-point phase noise correlation formulas
  - Resonator
  - Delay line
  - Delay lines can be quieter
- Given coating noise formulas
  - Relative magnitudes between coating and substrate noises.
  - Their behaviors against the beam size and other optical parameters

Proposed Activities

- Studies of intrinsic noises of coated mirrors
  - Thermal noise
  - Thermo-elastic noise
  - Relative significance of coating noise
  - Realistic mirror shapes

Broader Impacts

- Education
  - Student involvement through the collaboration with Stanford Group
- Contribution to LSC
  - The thermal noise estimation methodology
  - Impact on sorting out advanced optics designs
  - Impact on mirror material selection
- Impacts on other federally funded programs
  - Through advancement of computational physics methodology
    - NSF Industry/University Cooperative Research Center program
    - DOE/NERI project; “On-line NDE for advanced reactors”